Simply Connected Constant Mean Curvature Surfaces in $\mathbb{H}^2 \times \mathbb{R}$

Barbara Nelli - Harold Rosenberg

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Abstract

We prove that for $H > \frac{1}{\sqrt{3}}$ there is no properly embedded H-surface in $\mathbb{H}^2 \times \mathbb{R}$ with finite topology and one end.

1 Introduction

In [9], Meeks proved that if M is a properly embedded simply connected surface of constant mean curvature $H \neq 0$ in $\mathbb{R}^3$, then M is a round sphere. In particular, M can not be topologically $\mathbb{R}^2$. More generally, he proved there is no properly embedded H-surface of finite topology in $\mathbb{R}^3$, with exactly one end. Afterwards, in [7], a different proof of Meeks' Theorem was found and, in [6], it was extended to the hyperbolic space $\mathbb{H}^3$.

In this paper we consider this problem in $\mathbb{H}^2 \times \mathbb{R}$. There are properly embedded H-surfaces in $\mathbb{H}^2 \times \mathbb{R}$ that are topologically $\mathbb{R}^2$; there are entire graphs (vertical graphs over $\mathbb{H}^2$) for each H, $0 \leq H \leq \frac{1}{2}$ (see [10], [11]). We will prove that such a surface can not exist for $H > \frac{1}{\sqrt{3}}$. More generally, we prove:

Theorem 1.1 *For $H > \frac{1}{\sqrt{3}}$, there is no properly embedded H-surface in $\mathbb{H}^2 \times \mathbb{R}$ with finite topology and one end.*

Hsiang and Hsiang showed that any compact H-surface embedded in $\mathbb{H}^2 \times \mathbb{R}$ is a rotational sphere and it has mean curvature greater than $\frac{1}{2}$ (see [5] and [10]). Abresch and Rosenberg proved that, if the surface is simply connected, the same result holds for compact H-surfaces immersed in $\mathbb{H}^2 \times \mathbb{R}$ (see [1]).

It is interesting to consider to what extent Theorem 1.1 holds in other homogeneous 3-manifolds (for some other constant than $\frac{1}{\sqrt{3}}$). In $\mathbb{S}^2 \times \mathbb{R}$, there is no properly embedded H-surface with one end. To see this, notice that an end of such a surface M would have to go up, or down (but not both), since M is proper. So one can assume M is bounded below say by height zero. Then, do Alexandrov reflection in the "planes"

$\mathbb{S}^2 \times \{t\}$ coming up from $t = 0$, to conclude that the part of M below any $M \times \{t\}$, is a vertical graph. This contradicts the height estimates for such graphs (see [4]). So no such M exists in $\mathbb{S}^2 \times \mathbb{R}$.

The other homogeneous 3-manifolds (beside the space forms) are the Berger spheres, Heisenberg space and $\widetilde{PSL}(2,\mathbb{R})$. Since the Berger spheres are compact, the question is interesting in the last two spaces: Heisenberg space and $\widetilde{PSL}(2,\mathbb{R})$.

Another interesting question in Heisenberg space is whether the only embedded compact H-surfaces are the rotational spheres of constant mean curvature.

Theorem 1.1 has the following straightforward consequence.

Corollary 1.1 *A simply connected H-surface properly embedded in $\mathbb{H}^2 \times \mathbb{R}$, $H > \frac{1}{\sqrt{3}}$ is a rotational sphere.*

We remark that Theorem 1.1 and Corollary 1.1 do not hold if $H \leq \frac{1}{2}$. In fact, for any $H \in (0, \frac{1}{2}]$ there exists an entire rotational vertical H-graph (cf. [10]).

Theorem 1.1 follows from the following fact.

Theorem 1.2 *Let $H > \frac{1}{\sqrt{3}}$ and let M be a properly embedded H-surface in $\mathbb{H}^2 \times \mathbb{R}$ with finite topology and one end. Then M is contained in a vertical cylinder of $\mathbb{H}^2 \times \mathbb{R}$.*

Our proof of Theorem 1.2 holds in $\mathbb{R}^3$, as well.

The proof of Theorem 1.2 depends on the key result of the Plane Separation Lemma (cf. Section 2). The analogue of this Lemma in $\mathbb{R}^3$ and $\mathbb{H}^3$ was proved in [9] and [6] respectively.

Theorem 1.2 in $\mathbb{H}^2 \times \mathbb{R}$ does not hold without the one end hypothesis. In fact, there are examples of constant mean curvature cylinders lying in the tubular neighborhood of a horizontal geodesic (cf. [8]).

We conjecture that Theorem 1.1 holds for $H > \frac{1}{2}$. The bound $H > \frac{1}{\sqrt{3}}$ seems due only to technical reasons.

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2 Four Key Lemmas

Let L be the stability operator for a H-surface in a three-manifold N. For the notion of stability, see [10] and [3].

Lemma 2.1 *Let M be a compact H-surface in a three manifold N^3, transverse to some Killing vector field of N^3. Then M is stable.*

Proof. The proof is analogous to that of Remark 2.1 in [10]. We do it for completeness. Let $\Psi_s : N^3 \to N^3$ be the family of isometries of the ambient space giving rise to the given Killing vector field. Orient M by the mean curvature vector. It is enough to prove that, for any compact domain $D \subset M$, the first eigenvalue of the stability operator L on D is $\lambda \geq 0$. By contradiction, assume that $\lambda < 0$ and let f be the first eigenfunction. Then, $Lf = -\lambda f$, $f|\partial D = 0$ and one can assume $f|int(D) > 0$. Let Φ_t be the variation of D such that $< \frac{d\Phi_t}{dt}_{|t=0}, n >= f$, where n is the unit normal vector field orienting M. The first variation of the mean curvature for the normal variation fn is given by

$$\dot{H}(0)f = Lf = -\lambda f.$$

Hence, for positive small t and for any $x \in int(D)$, $\dot{H}(0)f(x) > 0$, and the mean curvature of $\Phi_t(D)$ at x is greater than H. Now, move D by Ψ_s, such that $\Psi_s(D) \cap \Phi_t(D) = \emptyset$. Then, come back by moving D by Ψ_s, in the opposite direction: at the first contact point between $\Psi_s(D)$ and $\Phi_t(D)$, the mean curvature of $\Psi_s(D)$ is smaller than the mean curvature of $\Phi_t(D)$, but $\Psi_s(D)$ lies in the mean convex side of $\Phi_t(D)$. This is a contradiction.

□

The following Lemma was proved in [10].

Lemma 2.2 (Distance Lemma) *Let M be a stable H-surface in $\mathbb{H}^2 \times \mathbb{R}$ with $H > \frac{1}{\sqrt{3}}$. Then, for any $p \in M$*

$$dist_M(p, \partial M) < \frac{2\pi}{\sqrt{3(3H^2-1)}} \tag{1}$$

Remark 2.1 If a distance lemma were true for $H > \frac{1}{2}$, then Theorem 1.1 would follow for $H > \frac{1}{2}$.

Abresh and Rosenberg generalized Lemma 2.2 to the case of an ambient manifold that is homogeneously regular (cf. [2]).

Lemma 2.3 *Let P be a vertical plane in $\mathbb{H}^2 \times \mathbb{R}$. Let S be a compact embedded H-surface with boundary on P and $H > \frac{1}{\sqrt{3}}$. Then the distance of S from P is bounded above by the constant $c_0 = \frac{2\pi}{\sqrt{3(3H^2-1)}}$.*

Proof. Let $p \in S$ be a furthest point from the plane P and let γ be a minimizing ambient geodesic in $\mathbb{H}^2 \times \mathbb{R}$ from the point p to the plane P. Denote by h the length of γ. Let $P(t)$ be the family of vertical planes, orthogonal to γ, obtained by translating P along γ by the isometry of $\mathbb{H}^2 \times \mathbb{R}$ which is translation along γ, parametrized such that $P(0) = P$ and $p \in P(h)$. We do Alexandrov reflection of S with the family $P(t)$, starting at $P(h)$. We conclude that the part of S on one side of $P(\frac{h}{2})$, say S^+, has no point where S is orthogonal to one of the planes $P(t)$, $\frac{h}{2} \leq t \leq h$. Hence the Killing vector field obtained from γ is transverse to S^+, since it is orthogonal to the family of planes $P(t)$. By Lemma 2.1, S^+ is stable. Hence, by the Distance Lemma, $h < c_0$.

□

In [5], Hsiang and Hsiang computed the explicit form of the profile curve of a rotational H-surface. The vertical diameter and the horizontal diameter of a rotational H-surface are $\frac{4H}{\sqrt{4H^2-1}}\tan^{-1}\frac{1}{\sqrt{4H^2-1}}$ and $2\sinh^{-1}\frac{4H}{4H^2-1}$, respectively. We will see that, in order to prove the Plane Separation Lemma, one needs that the minimum distance between the two vertical planes is the horizontal diameter of a rotational H-surface.

Lemma 2.4 (Plane Separation Lemma) *Let $H > \frac{1}{2}$ and let P_1 and P_2 be two disjoint vertical planes in $\mathbb{H}^2 \times \mathbb{R}$. Denote by P_1^+, P_2^+ the two disjoint half-spaces determined by these planes. Let $c_1 = 2\sinh^{-1}\frac{4H}{4H^2-1}$. If the distance between P_1 and P_2 is greater than c_1, then, for any properly embedded H-surface M with finite topology and one end, either $P_1^+ \cap M$ or $P_2^+ \cap M$ consists entirely of compact components.*

Proof. Assume by contradiction that both $P_1^+ \cap M$ and $P_2^+ \cap M$ contain non compact components. Then, there are two proper arcs $\alpha_1 : [0,\infty) \longrightarrow P_1^+ \cap M$ and $\alpha_2 : [0,\infty) \longrightarrow P_2^+ \cap M$.

As M has finite topology, the end of M is topologically an annulus and we can assume that both $\alpha_1(t)$ and $\alpha_2(t)$ lie in the annular end of M for t sufficiently large. Denote by $\alpha_1(0) = p_1$ and $\alpha_2(0) = p_2$. One can choose an embedded arc β on M from p_1 to p_2 such that the arc $\delta = \alpha_1 \cup \beta \cup \alpha_2$ bounds a simply connected domain on M (see Figure 1).

Let P be a vertical plane between P_1 and P_2 at equal distance from P_1 and P_2. Let B be a geodesic ball of $\mathbb{H}^2 \times \mathbb{R}$ containing β and let C be a circle in the plane P, such that

1. $C \cap B = \emptyset$, and B has non empty intersection with the disk in P bounded by C.

2. The tubular neighborhood T of C of radius $\frac{c_1}{2}$ is embedded and $T \cap B = \emptyset$.

We remark that T is contained in the closed slab between P_1 and P_2 (see Figure 2). Now, let B_1 be a geodesic ball containing $B \cup T$. There exist $x_1 \in \alpha_1 \setminus B_1$, $x_2 \in \alpha_2 \setminus B_1$ and an arc γ from x_1 to x_2, embedded in M, such that

1. $\gamma \cap B_1 = \emptyset$.

2. Denoting by ρ the sub-arc of δ between the points x_1 and x_2, then $\rho \cup \gamma$ is a simple closed curve with linking number ± 1 with the circle C.

3. $\rho \cup \gamma$ bounds a compact disk U on M.

Hence $T \cap U$ contains a disk E such that $\partial E \subset \partial T$ and the linking number between ∂E and the circle C is ± 1.

Now, let $\Pi : \tilde{T} \to T$ be the universal Riemannian covering space of T. E lifts to a compact disk $\tilde{E} \subset \tilde{T}$. Topologically T is $D^2 \times \mathbb{S}^1$ and E is isotopic to some $D^2 \times \{\text{point}\}$. Then, $\tilde{T}$ is topologically $D^2 \times \mathbb{R}$ and $\tilde{E}$ is isotopic to some $D^2 \times \{\text{point}\}$. In particular

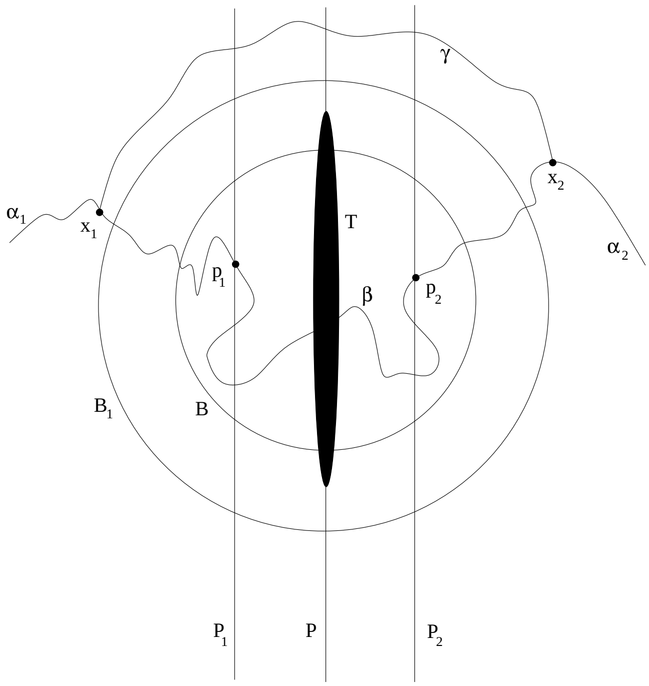

Figure 1

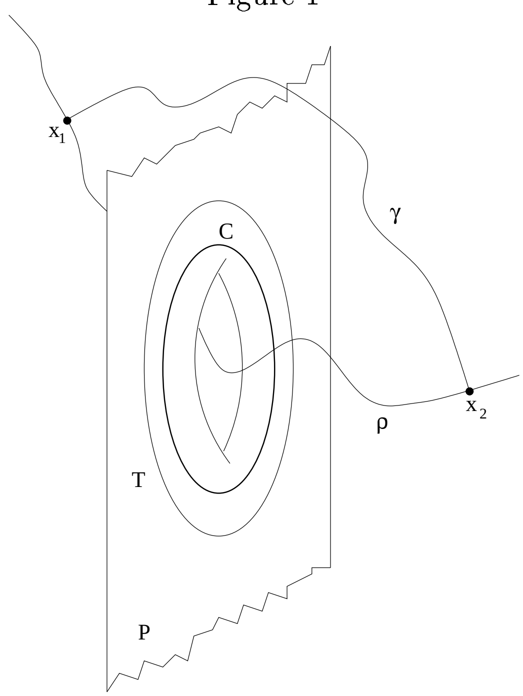

Figure 2

$\tilde{E}$ separates $\tilde{T}$ into two non compact connected components. Call $\tilde{W}$ the mean convex component.

Let $\tilde{C}$ be the curve in $\tilde{T}$ that projects to C by Π. For each point $p \in C$ there is a spherical H-surface, say $S_H(p)$, invariant by rotation about the vertical geodesic through p. It is clear that, if the radius of C is sufficiently large, then each $S_H(p)$ is contained in T. For any point $\tilde{p} \in \tilde{C}$, denote by $\tilde{S}_H(\tilde{p})$ the compact surface that projects to $S_H(p)$ by Π, where $\Pi(\tilde{p}) = p$.

One can find a point $\tilde{q}$ on $\tilde{C}$ such that $\tilde{S}_H(\tilde{q})$ is contained in $\tilde{W}$ and is disjoint from $\tilde{E}$. Then, move $\tilde{q}$ along $\tilde{C}$ towards $\tilde{E}$ till the first point $\tilde{q}_1$ such that $\tilde{S}_H(\tilde{q}_1)$ and $\tilde{E}$ are tangent. $\tilde{S}_H(\tilde{q}_1)$ is contained in $\tilde{W}$ and, at the tangent point, both $\tilde{E}$ and $\tilde{S}_H(\tilde{q})$ have curvature equal to H. So they should coincide, by the maximum principle. Contradiction.

$\square$

Remark 2.2 The Plane Separation Lemma holds (and the proof is the same) for P_1, P_2 horizontal planes. The distance between P_1 and P_2 must be larger than the vertical diameter of a rotational H-surface.

3 Cylindrically Bounded

Let us state again the cylindrically bounded Theorem.

Theorem 1.2 *Let $H > \frac{1}{2}$ and let M be a properly embedded H-surface in $\mathbb{H}^2 \times \mathbb{R}$ with finite topology and one end. Then M is contained in a vertical cylinder of $\mathbb{H}^2 \times \mathbb{R}$.*

Proof. We can assume that the point $\sigma = (\mathbf{0}, 0)$ belongs to M, $\mathbf{0}$ is an origin of $\mathbb{H}^2$. Let $\gamma : [0, r] \longrightarrow \mathbb{H}^2 \times \mathbb{R}$ be any horizontal geodesic, starting at σ, parametrized by arc-length and denote by $P(r)$ the vertical plane passing through $\gamma(r)$ orthogonal to γ. We claim that, there exists a constant c_2 (independent of the geodesic γ) such that, if $r > c_2$, then the half-space determined by $P(r)$ that does not contain the point σ is disjoint from M. This clearly implies that M is contained in the vertical cylinder with axis $\mathbf{0} \times \mathbb{R}$ and radius c_2.

Let us prove the claim. We choose $R > \max\{c_0, c_1\}$, where c_0 and c_1 are the constants given by the Lemmas 2.3 and 2.4, respectively. Denote by $P(R)^+$ the half-space determined by $P(R)$ containing the point σ and by $P(2R)^+$ the half-space determined by $P(2R)$ not containing the point σ. By the Plane Separation Lemma applied to the surface M, one has that one of the followings holds.

1. $M \cap P(R)^+$ has only compact components.

2. $M \cap P(2R)^+$ has only compact components.

If the former is true, then, by Lemma 2.3, the distance between the plane $P(R)$ and the point $\sigma \in M \cap P(R)^+$ must be at most c_0. This is a contradiction with our choice of R. Hence the latter is true. Then, again by Lemma 2.3, the maximum distance between $M \cap P(2R)^+$ and the plane $P(2R)$ is at most c_0, hence M is disjoint from the half-space determined by $P(2R + c_0)$ not containing the point σ.

So, choosing the constant $c_2 = 2\max\{c_0, c_1\} + c_0$, the claim is proved.

□

We now prove our main result.

Theorem 1.1 *For $H > \frac{1}{\sqrt{3}}$ there is no properly embedded H-surface in $\mathbb{H}^2 \times \mathbb{R}$ with finite topology and one end.*

Proof. Assume by contradiction that an H-surface M, satisfying the hypothesis, exists. Let $Q(t)$ be the horizontal geodesic plane at height t in $\mathbb{H}^2 \times \mathbb{R}$. By Theorem 1.2, M is contained in a vertical cylinder and, as M has only one proper end, M is bounded either above or below. We can assume that M is bounded below and that the lowest points of M lie in the plane $Q(0)$. As reflections with respect to the planes $Q(t)$ are isometries of $\mathbb{H}^2 \times \mathbb{R}$, we can apply the Alexandrov reflection method with the planes $Q(t)$ to the surface M. M is contained in a cylinder so, no accident can occur, moving $Q(0)$ up: there is no smallest t such that M becomes orthogonal to $Q(t)$ at some point; otherwise by Alexandrov, $Q(t)$ would be a plane of symmetry for M. Hence, for any $t > 0$, the part of M below $Q(t)$ is a vertical graph and there are no points of M below $Q(t)$, where M is orthogonal to one of the planes $Q(t)$. As $\frac{\partial}{\partial t}$ is a Killing vector field, by Lemma 2.1, the part of M below the plane $Q(t)$ is stable. But one can choose t larger than the constant of the Distance Lemma: a contradiction.

□

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Barbara Nelli
Dipartimento di Matematica Pura e Applicata, Universitá di L'Aquila
nelli@univaq.it
Harold Rosenberg
Institut de Mathématiques, Université Paris VII
rosen@math.jussieu.fr