Vertical Ends of Constant Mean Curvature $H = \frac{1}{2}$ in $\mathbb{H}^2 \times \mathbb{R}$

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Abstract

We prove existence of graphs over exterior domains of $\mathbb{H}^2 \times \{0\}$, *with constant mean curvature* $H = \frac{1}{2}$ *in* $\mathbb{H}^2 \times \mathbb{R}$, *provided the boundary curve satisfies some geometric conditions. Furthermore, we prove a vertical halfspace theorem for surfaces with constant mean curvature* $H = \frac{1}{2}$, *properly immersed in* $\mathbb{H}^2 \times \mathbb{R}$.

1 Introduction

In this paper we study vertical ends of surfaces with constant mean curvature $H = \frac{1}{2}$ in $\mathbb{H}^2 \times \mathbb{R}$, where a vertical end is a topological annulus, with no asymptotic point at finite height.

Our first result is the existence of vertical graphs, over exterior domains of $\mathbb{H}^2 \times \{0\}$, of constant mean curvature $H = \frac{1}{2}$, in $\mathbb{H}^2 \times \mathbb{R}$ (Theorem 3.1) provided that the boundary of the surface satisfies some geometric conditions (see Definition 3.1). Each graph is a vertical end. In fact, given a boundary data C, we solve a Dirichlet problem for the mean curvature equation $H = 1/2$ over an exterior domain of $\mathbb{H}^2 \times \{0\}$. The condition on the boundary C guarantees *a-priori* estimates for the gradient of solutions at the boundary.

In the proof of our existence Theorem, we use a classic fixed point theorem (see Theorem A.7 in Section 3). In order to apply it, we need to prove *a-priori* estimates for solutions of a family of Dirichlet problems. This is achieved by using rotational surfaces of constant mean curvature $H \leq \frac{1}{2}$ as geometric barriers.

Our existence result leads to the following conjecture.

Conjecture 1. *Every vertical graph end is asymptotic to a rotational graph end.*

If the conjecture were true, one could develop a theory of properly embedded vertical ends in $\mathbb{H}^2 \times \mathbb{R}$, analogous to the minimal ends theory in $\mathbb{R}^3$ (see [6], [16], [14]).

Then, we investigate about halfspace type results for surfaces of constant mean curvature $H = \frac{1}{2}$ in $\mathbb{H}^2 \times \mathbb{R}$. We are able to prove that a surface of constant mean curvature $H = \frac{1}{2}$, different from a rotational simply connected one, can not be properly immersed in the mean convex side of a simply connected rotational surface of constant mean curvature $H = \frac{1}{2}$ (see Theorem 4.1).

The mean curvature $\frac{1}{2}$ in $\mathbb{H}^2 \times \mathbb{R}$ plays the same role as the mean curvature zero in $\mathbb{R}^3$ and one in $\mathbb{H}^3$ (see [3]). In the discussion about vertical ends of constant mean curvature $H = \frac{1}{2}$ in $\mathbb{H}^2 \times \mathbb{R}$, some natural questions arise, inspired by the theory of minimal ends in $\mathbb{R}^3$ and of constant mean curvature one ends in $\mathbb{H}^3$. Before describing our results in this direction, let us recall some properties of minimal ends in $\mathbb{R}^3$. An embedded end of a minimal surface in $\mathbb{R}^3$, with finite total curvature is a graph of a function u defined on an exterior domain of the plane and has the following asymptotic behavior (see [16])

$$u(x, y) = a \ln(x^2 + y^2) + b + \frac{cx + dy}{x^2 + y^2} + O\left(\frac{1}{x^2 + y^2}\right) \tag{1}$$

The constant a is called the logarithmic growth of the end. The asymptotic behavior in (1) means that a final total curvature end of a minimal surface is asymptotic to a plane ($a = 0$) or to a catenoid ($a \neq 0$). See also [14], for an enlightening discussion about the ends of a minimal surface.

The end of a rotational surface of constant mean curvature $H = \frac{1}{2}$ in $\mathbb{H}^2 \times \mathbb{R}$ is the vertical graph of a function u_α of growth $\frac{1}{\sqrt{\alpha}}$ (see Definition 3.3 for the notion of growth). The asymptotic behavior of u_α is the following (see formula (5) in Section 2)

$$u_\alpha(\rho) = \frac{1}{\sqrt{\alpha}} e^{\frac{\rho}{2}} + \frac{3\alpha^2 - 1}{2\alpha^{\frac{3}{2}}} e^{-\frac{\rho}{2}} + k + O\left(e^{-\frac{3\rho}{2}}\right), \quad \rho \longrightarrow \infty$$

where $\alpha \in \mathbb{R}_+$ and $k \in \mathbb{R}$. A different way of stating Conjecture 1 is the following: a vertical graph end of mean curvature $H = \frac{1}{2}$ in $\mathbb{H}^2 \times \mathbb{R}$ has growth $\frac{1}{\sqrt{\alpha}}$ for some α. We are able to prove the following partial result (see Theorem 3.1): each vertical graph end obtained in Theorem 3.1 has weak growth $\frac{1}{\sqrt{\alpha}}$ for a certain α, i.e. it is weakly asymptotic to the end of a rotational surface of growth $\frac{1}{\sqrt{\alpha}}$ (see Definition 3.3 for the notion of weak growth).

An existence Theorem in $\mathbb{R}^3$, analogous to Theorem 3.1 is proved by Sa Earp and Toubiana (see [20]). We believe that an existence Theorem on exterior domains in $\mathbb{H}^2 \times \mathbb{R}$ holds with some necessary conditions on the boundary curve C (see [11] for an example in the Euclidean case).

In [4], the authors prove existence of complete surfaces in $\mathbb{H}^2 \times \mathbb{R}$, with constant mean curvature $H = \frac{1}{2}$ whose ends are vertical. Such surfaces are sisters of minimal surfaces in the Heisenberg group.

The paper is organized as follows. In Section 2, we describe the geometry of rotational ends of constant mean curvature $H \in (0, \frac{1}{2}]$ and we prove the Convex Hull Lemma. In Section 3 we prove the existence Theorem. Finally, in Section 4, we establish the halfspace Theorem.

2 Rotational Surfaces with $0 \leq H \leq \frac{1}{2}$

R. Sa Earp and E. Toubiana find explicit integral formulas for rotational surfaces of constant mean curvature $H \in (0, \frac{1}{2}]$ in [19]. A careful description of the geometry of these surfaces is contained in the Appendix of [13].

In this Section, we recall some properties of rotational surfaces of constant mean curvature $H \in (0, \frac{1}{2}]$ and we describe their asymptotic behavior.

Let $u : \Omega \longrightarrow \mathbb{R}$ be a C^2 function defined on a subset Ω of $\mathbb{H}^2 \times \{0\}$. The *vertical graph* of u is the subset of $\mathbb{H}^2 \times \mathbb{R}$ given by $\{(x, y, t) \in \Omega \times \mathbb{R} \mid t = u(x, y)\}$.

The vertical graph of a function $u : \mathbb{H}^2 \times \{0\} \longrightarrow \mathbb{R}$ has constant mean curvature H with respect to the upward normal vector field, if and only if u satisfies the following partial differential equation

$$\mathrm{div}_{\mathbb{H}} \left(\frac{\nabla_{\mathbb{H}} u}{W_u} \right) = 2H \tag{2}$$

where $\mathrm{div}_{\mathbb{H}}$, $\nabla_{\mathbb{H}}$ are the hyperbolic divergence and gradient respectively and $W_u = \sqrt{1 + |\nabla_{\mathbb{H}} u|^2_{\mathbb{H}}}$, being $|\cdot|_{\mathbb{H}}$ the norm in $\mathbb{H}^2 \times \{0\}$.

By abuse of notation, we will call graph of u, the vertical graph of u.

Denote by ρ the hyperbolic distance from the origin in $\mathbb{H}^2 \times \{0\}$. Then, the function whose graph is a rotational surface with constant mean curvature H must satisfies (cf. formula (21) in [19], with $l = 0$, $\alpha = -d$ and formula (9) in [13])

$$u^H_\alpha(\rho) = \int_{r^H_\alpha}^{\rho} \frac{-\alpha + 2H \cosh r}{\sqrt{\sinh^2 r - (-\alpha + 2H \cosh r)^2}} dr \tag{3}$$

where α is a real parameter and $r^H_\alpha = \operatorname{arccosh}\left(\frac{-2\alpha H + \sqrt{1-4H^2+\alpha^2}}{1-4H^2}\right)$ is the minimum such that $\sinh^2 r - (-\alpha + 2H \cosh r)^2 > 0$. The function u^H_α is defined up to an additive constant, that corresponds to a vertical translation of the rotational surface.

When $\alpha = 2H$, the function u^H_{2H} is defined on $\mathbb{H}^2 \times \{0\}$ and its graph is a simply connected rotational surface, denoted by S^H. For any $\alpha \neq 2H$, the graph of u^H_α is defined outside the disk D^H_α of radius r^H_α and it is vertical along the boundary of D^H_α. We choose the integration constant such that the graph of u^H_α is contained in the half-space $t \geq 0$, with boundary in the slice $t = 0$. Denote by $\mathcal{H}^H_\alpha$ the union of the graph of u^H_α with its symmetry with respect to the slice $t = 0$.

When $0 < \alpha < 2H$, the annuli $\mathcal{H}^H_\alpha$ are embedded, while for $\alpha > 2H$, they are immersed.

We are specially interested in the asymptotic behavior of the rotational surfaces. As the the asymptotic behavior for $H = \frac{1}{2}$, is quite different form the $H < \frac{1}{2}$ case, we analyze the two cases separately.

- $H = \frac{1}{2}$.

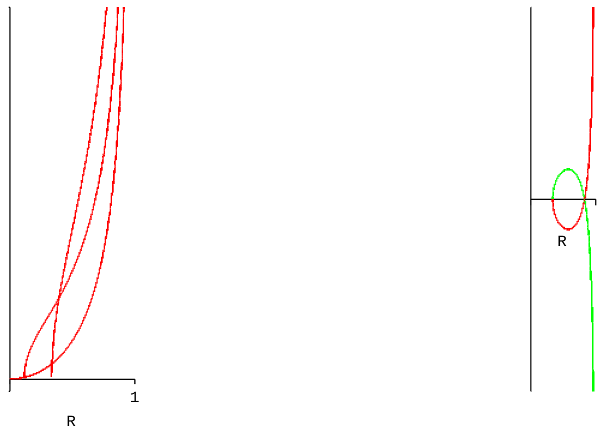

Figure 1: $H = \frac{1}{2}$: the profile curve in the embedded and immersed case ($R = \tanh \rho$).

For simplicity, we denote by u_α, D_α, r_α, $\mathcal{H}_\alpha$, S, the previous $u_\alpha^{\frac{1}{2}}$, $D_\alpha^{\frac{1}{2}}$, $r_\alpha^{\frac{1}{2}}$, $\mathcal{H}_\alpha^{\frac{1}{2}}$, $S^{\frac{1}{2}}$, respectively.

For any $\alpha \neq 1$, $\mathcal{H}_\alpha$ is a rotational annulus symmetric with respect to the plane $t = 0$. Replacing $H = \frac{1}{2}$ in formula (3) one has

$$u_\alpha(\rho) = \int_{-\ln \alpha}^{\rho} \frac{-\alpha + \cosh r}{\sqrt{2\alpha \cosh r - 1 - \alpha^2}} dr \tag{4}$$

where ρ is the hyperbolic distance from the origin (see Figure 1).

The radius of the disk D_α is $r_\alpha = -\ln \alpha$ and the function u_α is vertical along the boundary of D_α. Furthermore r_α is always greater or equal to zero, it is zero if and only if $\alpha = 1$ and tends to infinity as $\alpha \longrightarrow 0$. As we pointed out before, the graph of the function u_1 is entire and corresponds to the unique simply connected example.

Recall that we choose to compute the mean curvature of graphs, with respect to the upward unit normal. Then the unit normal to any $\mathcal{H}_\alpha$ points upward in the halfspace $t > 0$ (see Proposition 9 in [19] for a formula of the unit normal in terms of the function u_α).

A straightforward computation shows that the integrand function in (4) is equivalent to $\frac{1}{2\sqrt{\alpha}} e^{\frac{r}{2}} - c_\alpha e^{-\frac{r}{2}}$, for $r \longrightarrow \infty$, where c_α is a constant depending only on α. Then, by integrating, one has that $u_\alpha(\rho) \simeq \frac{1}{\sqrt{\alpha}} e^{\frac{\rho}{2}} + 2c_\alpha e^{-\frac{\rho}{2}} + k$, where k is the integration constant, for $\rho \longrightarrow \infty$.

By a more careful computation, one can prove that the asymptotic behavior of the function u_α is the following

$$u_\alpha(\rho) = \frac{1}{\sqrt{\alpha}} e^{\frac{\rho}{2}} + \frac{3\alpha^2 - 1}{2\alpha^{\frac{3}{2}}} e^{-\frac{\rho}{2}} + k + O\left(e^{-\frac{3\rho}{2}}\right), \quad \rho \longrightarrow \infty \tag{5}$$

We observe that, in the ball model, $\tanh \frac{\rho}{2} = R = \sqrt{x^2 + y^2}$, where (x, y) satisfies $x^2 + y^2 < 1$.

Now, it is very natural to give the following definition.

Definition 2.1 We define $\frac{1}{\sqrt{\alpha}}$ the *(exponential) growth* of the surface $\mathcal{H}_\alpha$.

As it is showed in Lemma 5.2 in [13], for $\alpha > 1$ the surfaces $\mathcal{H}_\alpha$ are immersed, while they are embedded for $\alpha \leq 1$. Then the growth of any immersed rotational surface is smaller than the growth of the simply connected surface S and the growth of any embedded rotational surface is greater than the growth of S.

Now, we describe how two rotational embedded surfaces $\mathcal{H}_\alpha$ and $\mathcal{H}_\beta$ intersect for $0 < \beta < \alpha < 1$. In this case, both ends of $\mathcal{H}_\beta$ are contained in the mean convex side of $\mathcal{H}_\alpha$. Furthermore, if one restricts to the half-space $t \geq 0$, then $\mathcal{H}_\alpha \cap \mathcal{H}_\beta \cap \{t \geq 0\}$ is a horizontal circle for every $0 < \alpha \neq \beta < 1$. For every $\alpha \in (0, 1)$, $S \cap \mathcal{H}_\alpha$ is a circle as well and, as $\alpha \longrightarrow 1$, the circle $S \cap \mathcal{H}_\alpha$ approaches the origin, while the upper end of $\mathcal{H}_\alpha$ approaches to the end of S.

Definition 2.2 *Let $0 < \beta < \alpha < 1$. We define horizontal distance between $\mathcal{H}_\alpha$ and $\mathcal{H}_\beta$ the distance between $\mathcal{H}_\alpha \cap \{t = 0\}$ and $\mathcal{H}_\beta \cap \{t = 0\}$, i.e. the positive number $d(\alpha, \beta) := r_\beta - r_\alpha = -\ln \frac{\beta}{\alpha}$.*

Next Lemma guarantees that the distance between $\mathcal{H}_\alpha \cap \{t = n\}$ and $\mathcal{H}_\beta \cap \{t = n\}$, when n is great, is almost the same as the distance at height zero. This result will be crucial in the barrier arguments in the proof of Theorem 3.1.

Lemma 2.1 *Let $0 < \beta < \alpha < 1$. Denote by R_α the radius of $\mathcal{H}_\alpha \cap \{t = n\}$ and by R_β the radius of $\mathcal{H}_\beta \cap \{t = n\}$ Then $R_\alpha \simeq 2 \ln n + \ln \alpha$, $R_\beta \simeq 2 \ln n + \ln \beta$, for $n \longrightarrow \infty$, that is $R_\beta - R_\alpha \simeq -\ln \frac{\beta}{\alpha}$, for $n \longrightarrow \infty$.*

Proof. It is a straightforward computation using the asymptotic behavior in (5) (see Figure 2).

□

It is clear that any surface obtained from $\mathcal{H}_\alpha$ either by a vertical translation or by a horizontal hyperbolic translation, has growth $\frac{1}{\sqrt{\alpha}}$. The effect of a vertical translation on formula (5) is obviously an additive constant. The image of $\mathcal{H}_\alpha$ by a horizontal hyperbolic translation intersects any slice in a circle. All such

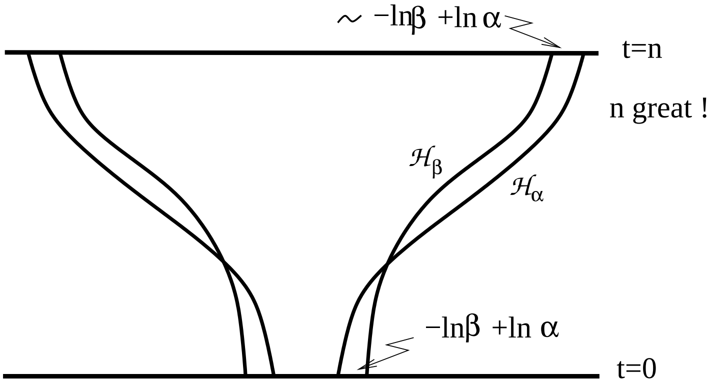

Figure 2: Distance between $\mathcal{H}_\alpha$ and $\mathcal{H}_\beta$.

circles have hyperbolic center on the same vertical geodesic, that is the image of the t-axis. Then, the translated surface has an asymptotic expansion as in (5), where ρ is the distance from the vertical geodesic, image of the t-axis.

In the following, we will refer to formula (5) for any surface obtained from $\mathcal{H}_\alpha$ either by a vertical translation or by a horizontal hyperbolic translation.

- $H < \frac{1}{2}$ (see Figure 3)

  For any $\alpha \neq 2H$, $\mathcal{H}_\alpha^H$ is a rotational annulus symmetric with respect to the plane $t = 0$.

  By formula (3), the radius of the disk D_α^H is $r_\alpha^H = \operatorname{arccosh}\left(\frac{-2\alpha H+\sqrt{1-4H^2+\alpha^2}}{1-4H^2}\right)$ and the function u_α^H is vertical at the boundary of D_α^H. Furthermore r_α^H is always greater or equal to zero, it is zero if and only if $\alpha = 2H$ and tends to infinity as $\alpha \longrightarrow \infty$. As we pointed out before, the graph of the function u_{2H}^H is entire and correspond to the unique simply connected example.

  Recall that we choose to compute the mean curvature of graphs, with respect to the upward unit normal. Then the unit normal to any $\mathcal{H}_\alpha^H$ points upward in the halfspace $t > 0$ (see Proposition 9 in [19] for a formula of the unit normal in terms of the function u_α^H).

  A straightforward computation shows that the integrand function in (3) is equivalent to $\frac{2H}{\sqrt{1-4H^2}} - c_{H,\alpha}e^{-r}$, for $r \longrightarrow \infty$, where $c_{H,\alpha}$ is a constant depending only on H and α. Then, by integrating, one has that

$$u_\alpha^H(\rho) \simeq \frac{2H}{\sqrt{1-4H^2}}\rho + c_{H,\alpha}e^{-\rho} + k \tag{6}$$

  where k is the integration constant, for $r \longrightarrow \infty$.

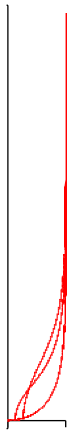

Figure 3: $H < \frac{1}{2}$: the profile curve in the embedded case.

As it is showed in Lemma 5.2 of [13], for $\alpha > 2H$ the surfaces $\mathcal{H}_\alpha^H$ are immersed, while they are embedded for $\alpha \leq 2H$. We notice that the parameter α does not appears in the leading term of the development of u_α^H. Then, the leading term of the asymptotic development of $\mathcal{H}_\alpha^H$ and $\mathcal{H}_\beta^H$ for $\alpha \neq \beta$ is the same (see Figure 5). It will be clear in the proof of our main result, how the different geometries of the families $\mathcal{H}_\alpha^H$ and $\mathcal{H}_\alpha$ will not allow to prove analogous results for the two families.

In spite of the global different behavior of the families $\mathcal{H}_\alpha$ and $\mathcal{H}_\alpha^H$, one has a nice relation between the two families on compact subsets of $\mathbb{H}^2 \times \mathbb{R}$. For any $\alpha > 0$, consider the two integrals (3) and (4). Notice that $\operatorname{arccosh}\left(\frac{-2\alpha H+\sqrt{1-4H^2+\alpha^2}}{1-4H^2}\right) \longrightarrow -\ln \alpha$, as $H \longrightarrow \frac{1}{2}$. Furthermore, for any fixed $\alpha > 0$ the integrand function in (4) converges pointwise to the integrand function in (3), as $H \longrightarrow \frac{1}{2}$. These two observations imply the following result.

Lemma 2.2 *For any fixed $0 < \alpha \neq 2H$, the function u_α^H converges to u_α on compact subset of $\mathbb{H}^2 \times \{0\} \setminus D_\alpha^H$, as $H \longrightarrow \frac{1}{2}$. Furthermore, u_{2H}^H converges to u_1 on compact subset of $\mathbb{H}^2 \times \{0\} \setminus D_{2H}^H$, as $H \longrightarrow \frac{1}{2}$.*

Figure 4 gives a idea of the convergence in the simply connected case. In order to understand better the convergence, we point out the following property. The radius r_α^H is increasing in H, when $\frac{\alpha}{2} < H < \frac{1}{2}$. Furthermore $r_\alpha^H \longrightarrow r_\alpha$ for $H \longrightarrow \frac{1}{2}$. Then, for H next to $\frac{1}{2}$, $r_\alpha^H < r_\alpha$, while the function u_α grows quicker than u_α^H. Hence, for N great enough, the radius of $\mathcal{H}_\alpha^H \cap \{t = N\}$ is greater than the radius of $\mathcal{H}_\alpha \cap \{t = N\}$.

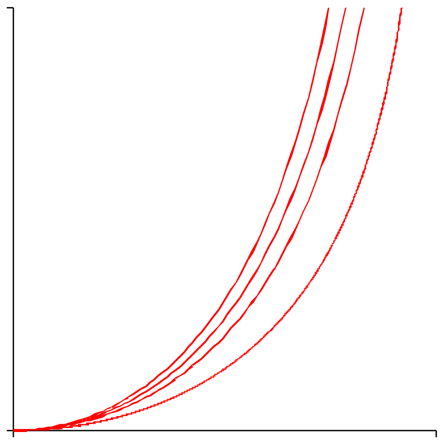

Figure 4: $H = \frac{1}{4}$, $H = \frac{1}{4^2}$, $H = \frac{1}{4^3}$, $H = \frac{1}{4^4}$.

In the following, we will refer to formula (6) for any surface obtained from $\mathcal{H}_\alpha^H$ either by a vertical translation or by a horizontal hyperbolic translation.

We close this section by proving a very interesting consequence of the existence of the rotational simply connected surfaces. Let K be a compact set in $\mathbb{H}^2 \times \mathbb{R}$. For any $H \in (0, \frac{1}{2}]$, we define $\mathcal{F}_K^H$ as follows. B belongs to $\mathcal{F}_K^H$ if the boundary ∂B is obtained by S^H either by vertical and horizontal translations or by symmetry with respect to a slice and K is contained in B.

Lemma 2.3 (Convex Hull Lemma) *(a) Let M be a compact surface immersed in $\mathbb{H}^2 \times \mathbb{R}$ with constant mean curvature $H \in (0, \frac{1}{2}]$. Then M is contained in the convex hull of the family $\mathcal{F}_{\partial M}^H$.*

(b)Let M be a compact surface immersed in $\mathbb{H}^2 \times \mathbb{R}$ with prescribed mean curvature function $H : M \longrightarrow (0, \frac{1}{2}]$. Then M is contained in the convex hull of the family $\mathcal{F}_{\partial M}^{\frac{1}{2}}$.

Proof. (a) Up to vertical translation, there exists a copy of S^H with the end on the top, containing M in its convex side. By abuse of notation, we denote by S^H any surface obtained by S^H by a hyperbolic isometry. Now, move S^H by a translation along some horizontal geodesic. If the first contact point p between S^H and M is an interior point of M, then the two surfaces are tangent at p and they have the same mean curvature vector at p. This is a contradiction by the maximum principle. Hence, one can move S^H horizontally until it touches ∂M. One can do the same for any horizontal geodesic and one can move S^H vertically as well. Furthermore, one can start with a surface with the end on the bottom and do the same proceeding. The result follows.

(b) The proof is analogous to the proof of (a).

□

The Convex Hull Lemma gives horizontal and vertical distance estimates in many geometric situations, for example in the proof of several uniqueness and symmetry results for surfaces with constant mean curvature in $\mathbb{H}^2 \times \mathbb{R}$, $H \leq \frac{1}{2}$ and boundary in two parallel horizontal planes [13].

3 Existence of Complete Graphs on Exterior Domains with $H = \frac{1}{2}$

Let Ω be a compact domain in $\mathbb{H}^2 \times \{0\}$. The subset $\mathbb{H}^2 \times \{0\} \setminus \Omega$ is called *exterior domain.* In this section, we prove the existence of graphs on exterior domains, provided the boundary satisfies some geometric conditions. It will be clear in the following that our conditions are very natural generalizations of the circle boundary case (see the examples after the statements of Theorem 3.1 and Theorem 3.2).

As we remarked in the introduction, an analogous result in $\mathbb{R}^3$ with geometrical assumption on the boundary curve, is proved by Sa Earp and Toubiana (see [20]). In [11], Kutev and Tomi proved existence and uniqueness of minimal graphs in $\mathbb{R}^3$, defined on exterior domains of $\mathbb{R}^2$, with boundary a curve satisfying some analytic conditions. The analogous exterior problem in $\mathbb{H}^3$ for mean curvature one is still open. As for the Dirichlet problem on compact domains in product spaces $M^n \times \mathbb{R}$ and in warped product spaces, it has been studied, for example, in [18] and [2] respectively.

Without loss of generality we can assume that the origin is inside Ω. Let $b > 0$ and denote by C_b the circle centered at the origin of radius b.

Definition 3.1 We say that Ω satisfies *the interior circle condition of radius b and distance d_b* if the following facts are satisfied

- b is such that for any point $p \in \partial\Omega$, $dist(p, C_b) < b$.
- For any $p \in \partial\Omega$, let γ_p be the geodesic between p and the origin: any translation of C_b along γ_p of ratio less or equal to $dist(p, C_b)$ is contained in Ω. Denote by $d_b = \max\limits_{p\in\partial\Omega} dist(p, C_b)$ (see Figure 5).

Notice that, if Ω satisfies the interior circle of radius b and distance d_b, then for any point $p \in \partial\Omega$ there exists a circle of radius b tangent to $\partial\Omega$ at p, contained in Ω (a translation of C_b). Furthermore, $d_b = 0$ if and only if $\partial\Omega$ is a circle.

Example 3.1 A non trivial example of domain Ω satisfying an interior circle condition is an Euclidean ellipse with small difference between the lengths of the axis. In the disk model for $\mathbb{H}^2 \times \{0\}$, consider an Euclidian ellipse E, centered at the origin with equation $\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$, where $0 < B < A < 1$, $A \simeq B$, $A < B\sqrt{2 - B^2}$. In this case, E satisfaies Definitiom 3.1 with $b = 2\text{arctanh}\,(B^2 A^{-1})$ and $d_b = 2\text{arctanh}(A) - 2\text{arctanh}\,(B^2 A^{-1})$.

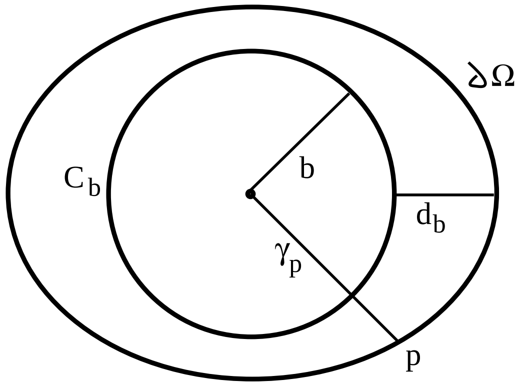

Figure 5: Interior circle condition.

It is clear that any small C^2 deformation of the ellipse E gives a domain satisfying Defnition 3.1 with the same b.

Before stating our results, let us give two further definitions.

Definition 3.2 Let E be the graph of a C^2 function u defined on the exterior domain $\mathbb{H}^2 \times \{0\} \setminus \Omega$. If $u_{|\partial\Omega}$ is bounded and $u(p) \longrightarrow \infty$ as p approaches the asymptotic boundary of $\mathbb{H}^2 \times \{0\}$, *i.e.* $\partial_\infty E$ is disjoint from $(\partial_\infty \mathbb{H}^2) \times \mathbb{R}$, we call E a *vertical graph end*. We also call *vertical graph end* the reflection of E with respect to a slice.

We notice that the ends of the rotational surfaces described in Section 2 are vertical graph ends, while there are many examples of graph ends of a constant mean curvature surface in $\mathbb{H}^2 \times \mathbb{R}$ that are not vertical (see [15]).

Equation (5) describes the asymptotic behavior of a rotational vertical graph end of exponential growth $\frac{1}{\sqrt{\alpha}}$. Now, we give the notion of growth for a general vertical graph end.

Definition 3.3 We say that a vertical graph end E of constant mean curvature $H = \frac{1}{2}$ has *growth* $\frac{1}{\sqrt{\alpha}}$ if the asymptotic behavior of the function whose graph is E is the same as in (5). We say that a vertical graph end E has *weak growth* $\frac{1}{\sqrt{\alpha}}$ if, for any $\varepsilon > 0$, there exists N_ε such that $E \cap \{t \geq N_\varepsilon\}$ is above some translation of $\mathcal{H}_{\alpha+\varepsilon}$ and below some translation of $\mathcal{H}_{\alpha-\varepsilon}$. By translation, we mean both vertical and horizontal.

The end of the simply connected surface S has growth one, while each end of an annulus $\mathcal{H}_\alpha$ has growth $\frac{1}{\sqrt{\alpha}}$.

Conjecture 1. *Every vertical graph end is asymptotic to a rotational graph end. If a vertical graph end of constant mean curvature $\frac{1}{2}$ has weak growth $H = \frac{1}{\sqrt{\alpha}}$, then it has growth $\frac{1}{\sqrt{\alpha}}$.*

As we observed in the Introduction, if the conjecture were true, one could develop a theory of vertical ends in $\mathbb{H}^2 \times \mathbb{R}$, analogous to the minimal ends theory in $\mathbb{R}^3$ (see [6], [16], [14]).

We have the following existence Theorems.

Theorem 3.1 *Let S be the simply connected rotational surface in $\mathbb{H}^2 \times \mathbb{R}_+$, tangent to $\mathbb{H}^2 \times \{0\}$ at the origin, with $H = \frac{1}{2}$. Let C be a C^3 simple closed curve contained in S such that the projection of C on $\mathbb{H}^2 \times \{0\}$ is one to one. Let Ω be the domain of $\mathbb{H}^2 \times \{0\}$ such that $\partial\Omega$ is the projection of C and assume that Ω contains the origin and satisfies an interior circle condition of radius b and distance d_b. Then, for any α such that $e^{d_b - b} < \alpha < 1$, there exists a complete graph on $\mathbb{H}^2 \times \{0\} \setminus \Omega$ with boundary C, $H = \frac{1}{2}$ and weak growth $\frac{1}{\sqrt{\alpha}}$.*

Theorem 3.2 *Let $\mathcal{H}_\gamma$ be the embedded rotational annulus in $\mathbb{H}^2 \times \mathbb{R}$ meeting the plane $t = 0$ orthogonally along a circle of radius $-\ln \gamma$, with $H = \frac{1}{2}$ and growth $\frac{1}{\sqrt{\gamma}}$. Let C be a C^3 simple closed curve contained in $\mathcal{H}_\gamma \cap \{t > 0\}$ such that the projection of C on $\mathbb{H}^2 \times \{0\}$ is one to one. Let Ω be the domain of $\mathbb{H}^2 \times \{0\}$ such that $\partial\Omega$ is the projection of C and assume that Ω contains the origin and satisfies an interior circle condition of radius b and distance d_b, with $b > -\ln \gamma$. Then, for any α such that $e^{d_b - b} < \alpha < \gamma$, there exists a complete graph on $\mathbb{H}^2 \times \{0\} \setminus \Omega$ with boundary C, $H = \frac{1}{2}$ and weak growth $\frac{1}{\sqrt{\alpha}}$.*

Remark 3.1 In Theorem 3.1, one can relax the hypothesis on the regularity of the boundary curve C to be continuous. In this case, the proof of existence makes use of the Perron's method. The Perron's method can be used also in the proof of Theorem 3.2.

Remark 3.2 It will be clear from the proof of Theorems 3.1 and 3.2 that the gradient of the solution u along the boundary curve C is bounded by a constant depending only on the geometry of C.

Let us describe some trivial examples of boundary curve C.

- Let C be a circle on S. It is straightforward that C satisfies an interior circle condition of radius b, for some b. Then, the family of rotational surfaces with boundary C and growth $\frac{1}{\sqrt{\alpha}}$, with $e^{-b} < \alpha < 1$, are solutions for Theorem 3.1. All such surfaces are contained in the mean convex side of S. This is a consequence of the geometry of the rotational surfaces and it can be seen by the maximum principle.

  Analogously if C is a circle on $\mathcal{H}_\gamma$ for some γ. Then C satisfies an interior circle condition of radius b, for some b. The family of rotational surfaces with boundary C and growth $\frac{1}{\sqrt{\alpha}}$ with $e^{-b} < \alpha < \gamma$ are solutions for Theorem 3.2. As before, all of them are contained in the mean convex side of $\mathcal{H}_\gamma$.

- Let C be the intersection between S and a horizontal translation of a $\mathcal{H}_\alpha$ (eventually $\alpha = 1$), along a geodesic δ through the origin. C is an analytic curve, because it is the intersection of two analytic surfaces. If the translation is small enough, the curve C satisfies the hypothesis of Theorem 3.1 and it bounds the two vertical ends given by S and the translation of $\mathcal{H}_\alpha$. The projection of C on the horizontal plane has at least one symmetry with respect to the vertical plane P_δ above the geodesic δ. S and $\mathcal{H}_\alpha$ are symmetric with respect to P_δ, as well. One obtains an example analogous to the last one, when C is the intersection between $\mathcal{H}_\beta$ and a horizontal translation of $\mathcal{H}_\alpha$ along a geodesic δ through the origin ($\alpha = \beta$ is allowed).

- C can be any curve on the surface S whose projection on $\mathbb{H}^2 \times \{0\}$ is a small C^2 deformation of the ellipse in Example 3.1.

Proof of Theorem 3.1. The proof of Theorem 3.1 is long, so we start by giving an idea of it.

Let ϕ be the function that describes the curve C as a graph on $\partial\Omega$. In order to prove the existence in Theorem 3.1, we have to solve the following Dirichlet problem

$$\begin{cases} \operatorname{div}_{\mathbb{H}}\left(\frac{\nabla_{\mathbb{H}} u}{W_u}\right) = 1 \ \text{ in } \ \mathbb{H}^2 \times \{0\} \setminus \Omega \\ \\ u = \phi \ \text{ on } \ \partial\Omega \end{cases} \tag{7}$$

We solve it by constructing a sequence of surfaces with mean curvature $H = \frac{1}{2}$, each of them being the graph of a function u_n on an annulus A_n whose interior boundary is $\partial\Omega$ and whose exterior boundary is a circle γ_n. The function u_n takes values ϕ on $\partial\Omega$ and n on γ_n. Then, we let n go to infinity and we prove that the sequence u_n converges to a solution of (7). The key point is that we succeed in obtaining *a-priori* height estimates and *a-priori* gradient estimates along the boundary for the solutions of (7), by using geometric barriers. Interior *a-priori* gradient estimates for the solutions of (7) are inferred as in [9], [17], [18].

In the proof of the Theorem 3.1, $\mathcal{H}_\alpha$ and $\mathcal{H}_\alpha^H$ will denote $\mathcal{H}_\alpha \cap \{t \geq \tau\}$ and $\mathcal{H}_\alpha^H \cap \{t \geq \tau\}$ respectively, translated down by τ, where $\tau > 0$ is arbitrarily small. We make this choice because we need graphs that are not vertical at the boundary, in order to use them as boundary barrier for the gradient.

We start by constructing the circle γ_n. This is a little tricky, but the motivation of our choice will be clear later.

Consider the surface $\mathcal{H}_\beta$ with $\beta = e^{-b}$. By Lemma 2.1, for any $\alpha > e^{d_b - b}$, the surface $\mathcal{H}_\alpha$ has the following property: the horizontal distance at height n between $\mathcal{H}_\beta$ and $\mathcal{H}_\alpha$ is almost $d(\alpha, \beta) := -\ln\beta + \ln\alpha > d_b$, for n great enough. Define $\Gamma_n = \mathcal{H}_\alpha \cap \{t = n\}$ and let γ_n be the projection of Γ_n on the plane $t = 0$. Notice that, this choice of α allows us to translate horizontally $\mathcal{H}_\beta$ by a distance d_b without touching Γ_n with $\mathcal{H}_\beta \cap \{t = n\}$.

Each u_n must satisfy

$$\begin{cases} \operatorname{div}_{\mathbb{H}}\left(\frac{\nabla_{\mathbb{H}} u}{W_u}\right) = 1 \ \text{ in } \ A_n \\ \\ u_n = \phi \ \text{ on } \ \partial\Omega \\ \\ u_n = n \ \text{ on } \ \gamma_n \end{cases} \tag{8}$$

The proof of Theorem 3.1 is in four steps.

STEP 1. *Existence of a $C^{2,\alpha}$ solution u_n of (8), for n large enough.*

STEP 2. *For any compact subset K of $\mathbb{H}^2 \times \{0\}$, existence of a uniform $C^{2,\alpha}$ bound on K for the sequence u_n.*

STEP 3. *Existence of a $C^{2,\alpha}$ solution u of (7).*

STEP 4. *Proof of the growth property for the solution u.*

Proof of the STEP 1. *Existence of a $C^{2,\alpha}$ solution u_n of (8), for n large enough.*

In order to prove existence of a $C^{2,\alpha}$ solution u_n of (8), we apply a fixed point Theorem (see Theorem 10.4 in [5] and Theorem A.7 in [1]). Our statement is slightly more general that Theorem A.7 in [1], but the proof is analogous. We will state Theorem A.7 in [1], later.

Roughly speaking, we construct a family of Dirichlet problems $\mathcal{D}_\sigma$, $\sigma \in [\sigma_0, 1]$, where σ_0 is a positive constant to be fixed later. For any $\sigma \in [\sigma_0, 1)$, the graph of the solution of $\mathcal{D}_\sigma$ has, as lower boundary, a circle C_σ lying on the simply connected rotational surface $S^{\frac{\sigma}{2}}$, C_σ converging to the curve C, for $\sigma \longrightarrow 1$ and, as upper boundary, the curve Γ_n lowered at height $f_n(\sigma) \le n$. The solution for $\sigma = \sigma_0$ is the part of $S^{\frac{\sigma_0}{2}}$ bounded by C_{σ_0} and $(\Gamma_n \times \mathbb{R}) \cap S^{\frac{\sigma_0}{2}}$. The solution for $\sigma = 1$ is the desired solution for (8).

By Theorem A.7 in [1], a solution of (8) exists, provided we find C^1 *a-priori* estimates for the solutions of the family $\mathcal{D}_\sigma$, $\sigma \in [\sigma_0, 1]$. That is: C^0 estimates, gradient estimates along the boundary and purely interior gradient estimates.

For the sake of clearness, before constructing the family $\mathcal{D}_\sigma$ of Dirichlet problems and proving the *a-priori* estimates for it, we prove the *a-priori* estimate for the case $H = \frac{1}{2}$ i.e. for the solution of (8). Then it will be easier to understand the construction of the family $\mathcal{D}_\sigma$ of Dirichlet problems and the relative estimates.

- *A-priori* C^0 estimates for u_n (see Figure 6).

  Denote by M_n the graph of u_n. By construction, the upper boundary Γ_n of M_n and the lower boundary C of M_n, lie in the non mean convex side of $\mathcal{H}_\beta$. Then, by the maximum principle M_n is entirely contained in the non mean convex side of $\mathcal{H}_\beta$ i.e. M_n is below $\mathcal{H}_\beta$. As the boundary of any M_n is contained in the mean convex side of S, the convex hull lemma guarantees that M_n lies above S.

  Remark 3.3 We notice that our C^0 *a-priori* estimates for u_n are independent of n.

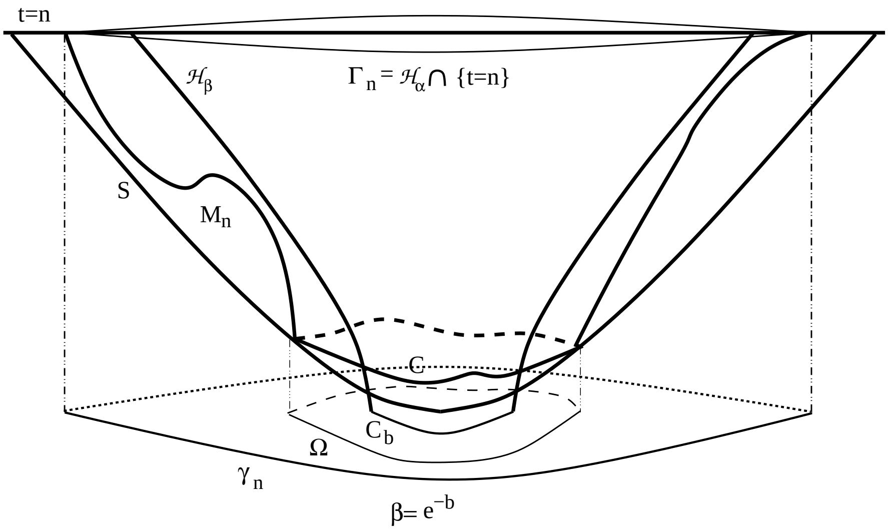

Figure 6: *A-priori* C^0 estimates for u_n

- *A-priori* estimates for the gradient of u_n along the boundary C (see Figure 7).

  As we proved in the previous step, M_n lies above S. Hence S itself bounds the normal derivative of each u_n from below along the curve C.

  By abuse of notation we denote by $\mathcal{H}_\beta$ any horizontal translation of $\mathcal{H}_\beta$. Notice that the intersection of $\mathcal{H}_\beta$ with the plane $t = 0$ is contained in Ω. Consider $p \in C$ and denote by $\mathcal{H}_\beta(p)$ the vertical translation of $\mathcal{H}_\beta$ at height $x_3(p)$. $\mathcal{H}_\beta(p)$ is disjoint from S and the radius of the intersection of $\mathcal{H}_\beta(p)$ with the plane $t = n$ is smaller than the radius of the intersection of $\mathcal{H}_\beta$ with the same plane. Then we translate horizontally $\mathcal{H}_\beta(p)$ till the horizontal projection of $\mathcal{H}_\beta(p) \cap \{t = x_3(p)\}$ and of C on the slice $t = 0$ are tangent at the projection of p on such slice. Notice that, before reaching this last position, the projection of $\mathcal{H}_\beta(p) \cap \{t = x_3(p)\}$ on the slice $t = 0$ stays inside Ω. Hence those $\mathcal{H}_\beta(p) \cap \{t = x_3(p)\}$ can not touch S. Lemma 2.1 guarantees that one does not touch Γ_n with $\mathcal{H}_\beta(p)$ and the maximum principle guarantees that M_n stays below $\mathcal{H}_\beta(p)$. Then $\mathcal{H}_\beta(p)$ is a good barrier for bounding the normal derivative of u_n at p and gives a bound for the normal derivative of u_n at p from above.

  Hence, there exists a positive constant L_p depending on $\mathcal{H}_\beta$ and S such that

$$\sup_{p \in C} |\nabla u_n(p)| \leq \sup_{p \in C} L_p < \infty \tag{9}$$

Remark 3.4 Notice that surface $\mathcal{H}_\beta$, that gives an estimates from above the gradient of u_n along C, depends only on the projection $\partial\Omega$ of C on the plane

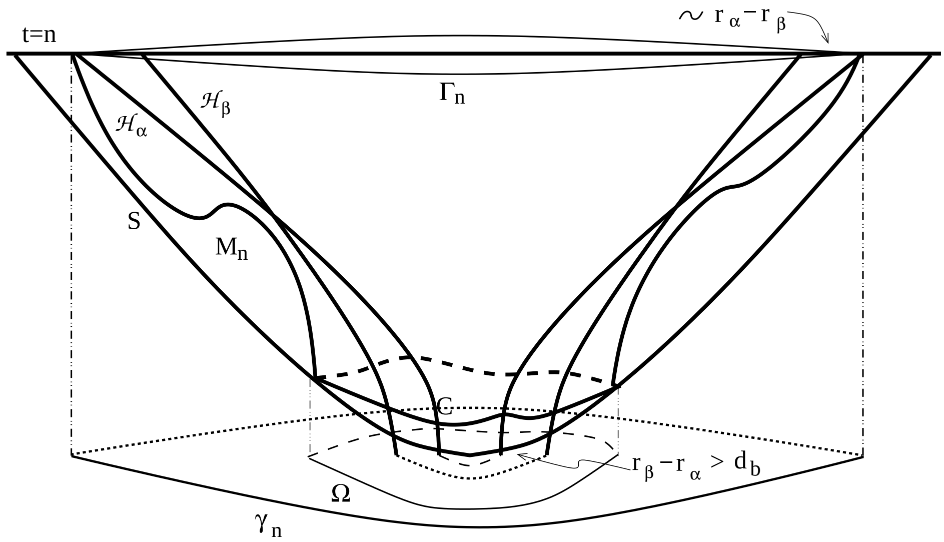

Figure 7: *A-priori* estimates for the gradient of u_n along the boundary C .

$t = 0$. Furthermore, the intersection of $\mathcal{H}_\beta$ with a horizontal plane is a circle with radius increasing with the height of the horizontal plane. Hence, if the upper boundary of M_n were the projection of Γ_n at a height lower than n, the estimate would yield as well, because such upper boundary would continue to be in the non mean convex side of $\mathcal{H}_\beta$.

- *A-priori* estimates for the gradient of u_n along the boundary Γ_n.

  Consider the slab $B = \{\min_{p \in C} x_3(p) \leq t \leq n\}$. By the maximum principle the graph of u_n is contained in the slab B, hence it is below the slice containing the boundary curve Γ_n. Now let p be any point of Γ_n. By the geometry of S, it is clear that one can find a translation of S (almost vertical in the slab B) tangent to Γ_n at the point p and containing M_n in its mean convex side. Such translation of S bounds the normal derivative of u_n at p from below.

  Hence, there exists a positive constant K_p such that

$$\sup_{p \in \Gamma_n} |\nabla u_n(p)| \leq \sup_{p \in \Gamma_n} K_p < \infty \tag{10}$$

Remark 3.5 If the upper boundary were the projection of Γ_n at a height $n_1 < n$, the previous estimate at the corresponding point p of the boundary, holds with the same S as before translated vertically by n_1.

- Interior *a-priori* estimates on the gradient of the solution u_n.

Without loss of generality we can assume that our solution are in the class $C^3(\Omega)$.

Theorem 1.1 in [18] guarantees purely interior gradient bounds at any point $p \in \Omega$, for any $C^3(\Omega)$ solution of (8) depending on the $C^0(\Omega)$ bounds and on the distance of p from the boundary (see formula 1.3 in [18]). Then, Theorem 3.1 in [18] guarantees global *a-priori* gradient bounds in Ω for any $C^3(\Omega)$ solution of (8), depending only on the height and on the boundary gradient bound.

The interior gradient bound can be obtained from [17] as well.

So, we have completed the *a-priori* estimates for the solution of the Dirichlet problem (8).

Now, we construct the family $\mathcal{D}_\sigma$ of Dirichlet problems, $\sigma \in [\sigma_0, 1]$. First, we define σ_0. By by Lemma 2.2, for any fixed n, $\mathcal{H}_\beta^{\frac{\sigma}{2}}$ converges to $\mathcal{H}_\beta$, in the slab $0 \le t \le n$, as $\sigma \longrightarrow 1$. Let $\delta = \frac{d(\alpha,\beta)-d_b}{2}$, where $d(\alpha, \beta)$ is defined in Definition 2.2. There exists σ_0 depending on δ and n such that, for any $\sigma \in [\sigma_0, 1]$:

(a) the distance between $\mathcal{H}_\beta^{\frac{\sigma}{2}} \cap \{t = h\}$ and $\mathcal{H}_\beta \cap \{t = h\}$ for $h \in [0, n]$, is smaller than δ;

(b) for any vertical geodesic γ intersecting both $\mathcal{H}_\beta^{\frac{\sigma}{2}}$ and $\mathcal{H}_\beta$, the points $\gamma \cap \mathcal{H}_\beta^{\frac{\sigma}{2}}$ and $\gamma \cap \mathcal{H}_\beta$ have distance smaller than δ.

As δ depends only on the geometry of the curve C, so does σ_0 for any fixed n.

The lower boundary of the graph of the solution of $\mathcal{D}_\sigma$ is constructed as follows.

Consider the vertical cylinder over $\partial\Omega$ and, for any $\sigma \in [\sigma_0, 1]$, denote by C_σ its intersection with the surface $S^{\frac{\sigma}{2}}$. Let ϕ_σ be the unique function on $\partial\Omega$ such that the graph of ϕ_σ is the curve C_σ. Our choice of the boundary curve C_σ is motivated by the fact that it lies in $S^{\frac{\sigma}{2}}$. Then, we can use $S^{\frac{\sigma}{2}}$ as barrier for bounding the gradient of solutions of $\mathcal{D}_\sigma$ from below.

The upper boundary of the graph of the solution of $\mathcal{D}_\sigma$ is constructed as follows.

Consider the vertical cylinder over γ_n, and intersect it with $S^{\frac{\sigma_0}{2}}$. Denote by h_0 the height of the horizontal circle $(\gamma_n \times \mathbb{R}) \cap S^{\frac{\sigma_0}{2}}$.

Define an increasing, C^2 function $f_n : [\sigma_0, 1] \longrightarrow [h_0, n]$ such that $f_n(\sigma_0) = h_0$ and $f_n(1) = n$ and that $f_n(\sigma)$ is greater than the third coordinate of the intersection $(\Gamma_n \times \mathbb{R}) \cap S^{\sigma/2}$, for any $\sigma \in (\sigma_0, 1)$.

Consider the following family $\mathcal{D}_\sigma$ of Dirichlet problems, $\sigma \in [\sigma_0, 1]$.

$$\mathcal{D}_\sigma = \begin{cases} F[u_n^\sigma] = \operatorname{div}\left(\frac{\nabla u_n^\sigma}{W_{u_n^\sigma}}\right) = \sigma \ \text{ in } \ A_n \\ \\ u_n^\sigma = \phi_\sigma \ \text{ on } \ \partial\Omega \\ \\ u_n^\sigma = f_n(\sigma) \ \text{ on } \ \gamma_n \end{cases} \tag{11}$$

Theorem (see Theorem A.7 in [1]). *The Dirichlet problem (8) has a $C^{2,\alpha}(\overline{A}_n)$ solution if there exists a constant C, independent of σ, such that every $C^{2,\alpha}(\overline{A}_n)$ solution u_n^σ of* (11)*, for any $\sigma \in [\sigma_0, 1]$, satisfies*

$$||u_n^\sigma||_{C^1(\overline{A}_n)} \leq C \tag{12}$$

By the previous Theorem, Step 1 is achieved as soon as we prove the C^1 estimate (12). In order to prove such *a-priori* estimate, we proceed as in the case of mean curvature $H = \frac{1}{2}$

- *A-priori* C^0 estimates on the solutions of (11).

  Denote by M_n^σ the graph of u_n^σ. For any $\sigma \in [\sigma_0, 1]$, the lower boundary C_σ, of M_n^σ lies on $S^{\frac{\sigma}{2}}$, while the upper boundary, that is the circle $(\Gamma_n \times \mathbb{R}) \cap \{t = f_n(\sigma)\}$, lies in the mean convex side of $S^{\frac{\sigma}{2}}$. Then by the maximum principle, M_n lies above $S^{\frac{\sigma}{2}}$, hence above $S^{\frac{\sigma_0}{2}}$. Furthermore, by the maximum principle, each M_n^σ lies below the plane $t = n$. So we get a priori height estimates independent of σ.

- *A-priori* estimates on the gradient of the solutions of (11) along C_σ.

  As M_n lies above $S^{\frac{\sigma}{2}}$, $S^{\frac{\sigma}{2}}$ itself bounds the normal derivative of each u_n from below along the curve C.

  For any $\sigma \in [\sigma_0, 1]$, consider the surface $\mathcal{H}_\beta^{\frac{\sigma}{2}}$. By abuse of notation, we denote any horizontal translation of $\mathcal{H}_\beta^{\frac{\sigma}{2}}$ by $\mathcal{H}_\beta^{\frac{\sigma}{2}}$.

  The horizontal distance between $\mathcal{H}_\beta^{\frac{\sigma}{2}}$ and $\mathcal{H}_\beta$ in the slab $0 \leq t \leq f_n(\sigma)$ is less than $\delta < \frac{d(\alpha,\beta) - d_b}{2}$, as $f_n(\sigma) \leq n$, i.e. $\mathcal{H}_\beta^{\frac{\sigma}{2}}$ is close to $\mathcal{H}_\beta$ in the slab $0 \leq t \leq f_n(\sigma)$.

  Then, one proceeds as in the $H = \frac{1}{2}$ case. One can move horizontally $\mathcal{H}_\beta^{\frac{\sigma}{2}}$ and approach any point of C_σ with $\mathcal{H}_\beta^{\frac{\sigma}{2}}$ without touching the upper boundary of M_n^σ. The previous assertion depends on the fact that the upper boundary of M_n^σ lies at most at height n (see Remark 3.5).

  The maximum principle guarantees that M_n^σ stays below $\mathcal{H}_\beta^{\frac{\sigma}{2}}$ for any $\sigma \in [\sigma_0, 1]$. So we get a priori estimates on the gradient of the solutions u_n^σ along C_σ independent of σ, for $\sigma \in [\sigma_0, 1]$.

- *A-priori* estimates on the gradient of the solutions of (11) along the upper boundary of M_n^σ, $(\Gamma_n \times \mathbb{R}) \cap \{t = f_n(\sigma)\}$.

  Remark 3.5 and the fact that the mean curvature of S is greater than $\frac{\sigma}{2}$ for any $\sigma \in [\sigma_0, 1]$, yields that we can do the same reasoning as in the case of $H = \frac{1}{2}$, with the surface S itself.

- Interior *a-priori* estimates on the gradient of the solutions of (11).

  This is exactly the same as in $H = \frac{1}{2}$ case.

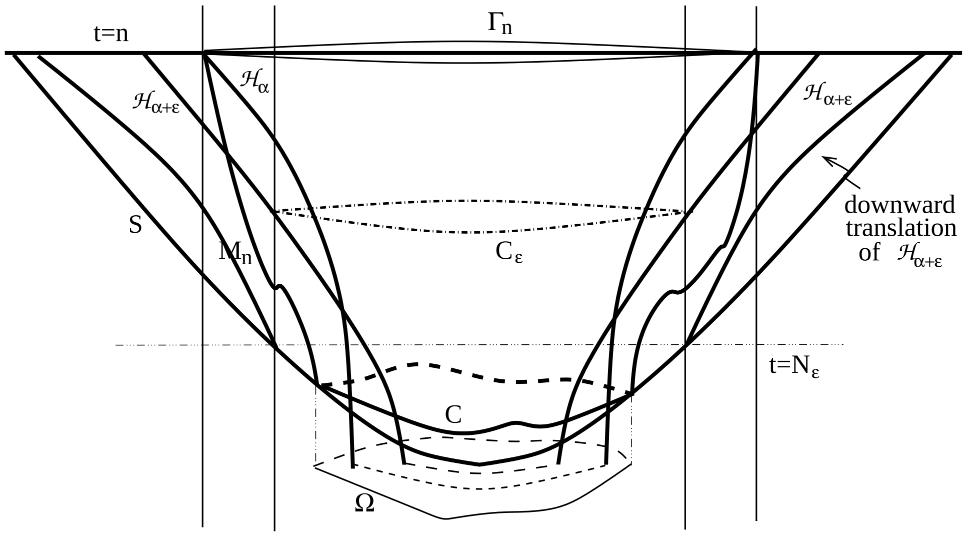

Figure 8: Growth Property.

Proof of STEP 2. *For any compact subset K of $\mathbb{H}^2 \times \{0\} \setminus \Omega$, existence of a uniform $C^{2,\alpha}$ bound on K for the sequence u_n.*

By standard elliptic theory, a C^1 *a-priori* bound on u_n yields a $C^{2,\alpha}$ *a-priori* bound on u_n. Hence, it is enough to prove that, for any compact subset K of $\mathbb{H}^2 \times \{0\}$, there is an uniform C^1 bound on the sequence u_n.

Let K be a compact subset of $\mathbb{H}^2 \times \{0\} \setminus \Omega$. For n large enough, $K \subset A_n$.

By Remark 3.3, the C^0 estimate found in Step 1 is independent on n, hence it yields a C^0 uniform bound for the family u_n on the subset K.

The interior gradient bound for u_n at a point p depends only on $u_n(p)$ and on the distance of p from the boundary (see formula 1.3 in [18]). This yields a uniform interior gradient bound for u_n on the subset K.

The interior gradient bound on the subset K can be obtained from [17] as well.

Proof of STEP 3. *Existence of a $C^{2,\alpha}$ solution u of (7).*

The annuli A_n exhaust the exterior domain $\mathbb{H}^2 \times \{0\} \setminus \Omega$, and, for any n, $u_{n|\partial\Omega} = \phi$. Hence STEP 2 and Ascoli-Arzela's theorem guarantee that the sequence u_n converges to a solution u of (7). Denote by M the graph of the function u.

Proof of STEP 4. *Proof of the growth property for the solution u (see Figure 8).*

Fix α between $e^{d_b - b}$ and one. By construction, each M_n lies below $\mathcal{H}_\alpha$. Hence so does the limit graph M. Now, fix $\varepsilon > 0$ such that $\varepsilon + \alpha < 1$ and consider $\mathcal{H}_{\alpha+\varepsilon}$. The intersection of $\mathcal{H}_{\alpha+\varepsilon}$ with $t = n$, for n great enough, is a circle with radius greater than the radius of Γ_n. Let C_ε be a circle in $\mathcal{H}_{\alpha+\varepsilon}$ with radius greater than the radius of $\mathcal{H}_\alpha \cap \mathcal{H}_{\alpha+\varepsilon}$, such that the projection of C_ε on the plane $\mathbb{H}^2 \times \{0\}$ bounds a domain containing Ω. Notice that such domain contains the projection on $\mathbb{H}^2 \times \{0\}$ of $\mathcal{H}_\alpha \cap \mathcal{H}_{\alpha+\varepsilon}$

as well. Denote by $(\mathcal{H}_{\alpha+\varepsilon})^+$ the vertical end of $\mathcal{H}_{\alpha+\varepsilon}$ bounded by C_ε. Translate $(\mathcal{H}_{\alpha+\varepsilon})^+$ downward until the boundary C_ε touches S. Let N_ε be the height of the translated of C_ε. By abuse of notation we continue to call $(\mathcal{H}_{\alpha+\varepsilon})^+$ any vertical translation of $(\mathcal{H}_{\alpha+\varepsilon})^+$.

We claim that M_n is above $(\mathcal{H}_{\alpha+\varepsilon})^+$ in the slab $[N_\varepsilon, n]$ for any n big enough.

Translate downward of the surface $(\mathcal{H}_{\alpha+\varepsilon})^+$ of $N_\varepsilon - n$. Notice that, after the translation, the boundary C_ε lies in the slice $t = 2N_\varepsilon - n$, while the previous $(\mathcal{H}_{\alpha+\varepsilon})^+ \cap \{t = n\}$ lies in the slice $t = N_\varepsilon$. Then $(\mathcal{H}_{\alpha+\varepsilon})^+$ does not intersect $M_n \cap [N_\varepsilon, n]$. In fact $M_n \cap [N_\varepsilon, n]$ is contained inside S while $(\mathcal{H}_{\alpha+\varepsilon})^+ \cap [N_\varepsilon, n]$ is outside S. We claim that we can translate $(\mathcal{H}_{\alpha+\varepsilon})^+$ upward until C_ε reaches the slice $t = N_\varepsilon$, without having any contact point with $M_n \cap [N_\varepsilon, n]$. By the maximum principle the first contact point cannot be interior. Furthermore the boundary C_ε cannot touch $M_n \cap [N_\varepsilon, n]$ because the former lies out of the mean convex side of S until it reaches the plane $t = N_\varepsilon$ while the latter lies in the mean convex side of S. Finally, $(\mathcal{H}_{\alpha+\varepsilon})^+$ can not touch the upper boundary Γ_n of M_n because, for any considered translation of $\mathcal{H}_{\alpha+\varepsilon}$, the radius of the circle $(\mathcal{H}_{\alpha+\varepsilon})^+ \cap \{t = n\}$ is strictly greater than the radius of Γ_n.

Hence the claim is proved.

As N_ε is independent of n, one has that the limit surface $M \cap \{t > N_\varepsilon\}$ lies above $(\mathcal{H}_{\alpha+\varepsilon})^+$. Then the weak growth of M is $\frac{1}{\sqrt{\alpha}}$.

This achieves the proof of Theorem 3.1.

$\square$

Proof of Theorem 3.2. The proof of Theorem 3.2 is analogous to the proof of Theorem 3.1 with the following changes.

- Replace S by $\mathcal{H}_\gamma$ and $S^{\frac{\sigma}{2}}$ by $H_\gamma^{\frac{\sigma}{2}}$
- By the mean convex side of $\mathcal{H}_\gamma$ ($\mathcal{H}_\gamma^{\frac{\sigma}{2}}$), we mean the intersection of the mean convex side of $\mathcal{H}_\gamma$ (respectively $\mathcal{H}_\gamma^{\frac{\sigma}{2}}$) with the halfspace $t \geq 0$.

For the sake of clearness, we point out that the further hypothesis $\alpha < \gamma$ guarantees that the upper boundary circle $\Gamma_n = \mathcal{H}_\alpha \cap \{t = n\}$ stays in the mean convex side of $\mathcal{H}_\gamma$.

$\square$

Let us state two easy consequences of the existence Theorems.

Corollary 3.1 *Let C be a simple closed curve contained in S satisfying the hypothesis of Theorem 3.1 If C has a symmetry with respect to some vertical plane through the origin, then each surface constructed in Theorem 3.1 has the same symmetry as C.*

Corollary 3.2 *Let C be a simple closed curve contained in a rotational annulus $\mathcal{H}_\gamma$ with $H = \frac{1}{2}$ and growth $\frac{1}{\sqrt{\gamma}}$, satisfying the hypothesis of Theorem 3.2. If C has a symmetry with respect to some vertical plane through the origin, then each surface constructed in Theorem 3.2 has the same symmetry as C.*

Proof of the Corollaries. We assume the notations of the proof of Theorem 3.1. As Γ_n is a circle centered at the origin, if C has a symmetry with respect to a vertical plane through the origin, then $C \cup \Gamma_n$ has the same symmetry as C. Fix n and consider the solution u_n of (8). By Alexandrov reflection method, the graph of u_n (and hence the function u_n) has the same symmetry as well. Then the limit function u of the sequence $\{u_n\}$ has the same symmetries as each u_n and so does its graph.

$\Box$

Conjecture 2. *Let Γ be a simple closed curve in $\mathbb{H}^2 \times \mathbb{R}$ that is the boundary of a graph M of growth $\frac{1}{\sqrt{\alpha}}$ over an exterior domain. Then M is the only graph with boundary Γ and growth $\frac{1}{\sqrt{\alpha}}$.*

Conjecture 3. *Let M be a surface with constant mean curvature $H = \frac{1}{2}$ properly embedded in $\mathbb{H}^2 \times \mathbb{R}$, with two vertical graph ends. Then M is a rotational surface.*

4 Vertical Halfspace Theorem

D. Hofmann e W. Meeks proved the Halfspace Theorem in [8]: there is no minimal surface properly immersed in a halfspace of $\mathbb{R}^3$. Halfspace theorem for minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ is false, in fact there are many minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ that have bounded third coordinates (see [12], [21]). It is natural to investigate about halfspace type results for surfaces of constant mean curvature $H = \frac{1}{2}$ in $\mathbb{H}^2 \times \mathbb{R}$. We are able to prove the following result.

Theorem 4.1 *Let S be a simply connected rotational surface with constant mean curvature $H = \frac{1}{2}$. Let Σ be a surface with constant mean curvature $H = \frac{1}{2}$, different from a rotational simply connected one. Then, Σ can not be properly immersed in the mean convex side of S.*

In [10] L. Hauswirth, H. Rosenberg and J. Spruck prove a halfspace type theorem for surfaces on one side of a horocylinder.

Proof. One can assume that the surface S is tangent to the slice $t = 0$ at the origin and it is contained in $\{t \geq 0\}$. Suppose, by contradiction, that Σ is contained in the mean convex side of S. Lift vertically S. If there is an interior contact point between Σ and the translation of S, one has a contradiction by the maximum principle. As Σ is properly immersed, Σ is asymptotic at infinity to a vertical translation of S. One can assume that the surface Σ is asymptotic to the S tangent to the slice $t = 0$ at the origin and contained in $\{t \geq 0\}$.

Let h be the height of one lowest point of Σ. Denote by $S(h)$ the vertical lifting of S of ratio h. One has one of the following facts.

- $S(h)$ and Σ has a first finite contact point p : this means that $S(h - \varepsilon)$ does not meet Σ at a finite point, for $\varepsilon > 0$ and then $S(h)$ and Σ are tangent at p with mean curvature vector pointing in the same direction. In this case, by the maximum principle $S(h)$ and Σ should coincide. Contradiction.

- $S(h)$ and Σ meet at a point p, but p is not a first contact point.Then, for ϵ small enough, $S(h-\varepsilon)$ intersect Σ transversally.

Denote by W the non compact subset of $\mathbb{H}^2 \times \mathbb{R}$ above S and below $S(h-\varepsilon)$.

It follows from the maximum principle that there are no compact component of Σ contained in W. Denote by Σ_1 a non compact component of Σ contained in W. Note that the boundary of Σ_1 is contained in $S(h-\varepsilon)$. Consider the family of rotational non embedded surfaces $\mathcal{H}_\alpha$, $\alpha > 1$. Translate each $\mathcal{H}_\alpha$ vertically in order to have the waist on the plane $t = h - \varepsilon$. By abuse of notation, we continue to call the translation, $\mathcal{H}_\alpha$. The surface $\mathcal{H}_\alpha$ intersects the plane $t = h - \varepsilon$ in two circles. Denote by ρ_α the radius of the larger circle. Denote by $\mathcal{H}_\alpha^+$, the part of the surface outside the cylinder of radius ρ_α. Notice that $\mathcal{H}_\alpha^+$ is embedded. By the geometry of the $\mathcal{H}_\alpha^+$, when α is great enough, say α_0, $\mathcal{H}_{\alpha_0}^+$ is outside the mean convex side of S (cf. [13]). Then, $\mathcal{H}_{\alpha_0}^+$ does not intersect Σ. Furthermore, when $\alpha \longrightarrow 1$, $\mathcal{H}_\alpha^+$ converge to $S(h-\varepsilon)$. Now, start to decrease α from α_0 to one. Before reaching $\alpha = 1$, the surface $\mathcal{H}_\alpha^+$ first meets S and then touches Σ_1 at an interior finite point. This depends on the following two facts.

- The boundary of Σ_1 lies on $S(h-\varepsilon)$ and the boundary of any of the $\mathcal{H}_\alpha^+$ lies on the horizontal plane $t = h - \varepsilon$.
- The growth of any of the $\mathcal{H}_\alpha^+$ is strictly smaller than the growth of S. Thus the end of $\mathcal{H}_\alpha^+$ is outside the end of S.

The existence of an interior contact point is a contradiction by the maximum principle.

$\Box$

Conjecture 4. *Let S be a simply connected rotational surface with constant mean curvature $H = \frac{1}{2}$. Let Σ be a surface with constant mean curvature $H = \frac{1}{2}$ different from a rotational simply connected one. Then, Σ can not be properly embedded in the non mean convex side of S with mean curvature vector pointing towards S.*

Conjecture 5. *Let Σ_1, Σ_2 be two properly embedded surfaces with constant mean curvature $H = \frac{1}{2}$, different from the rotational simply connected one. Then Σ_i can not lie in the mean convex side of Σ_j, $i \neq j$.*

Another statement for the same conjecture.

Conjecture 5′. *Let Σ be a properly embedded surface with constant mean curvature $H = \frac{1}{2}$, different from the rotational simply connected one. Then, there is no properly embedded surface with constant mean curvature $H = \frac{1}{2}$, contained in the mean convex side of Σ.*

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