Finiteness properties of pairs of 2×2 sign-matrices via real extremal polytope norms

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Abstract

This paper deals with the joint spectral radius of a finite set of matrices. We say that a set of matrices has the *finiteness property* if the maximal rate of growth, in the multiplicative semigroup it generates, is given by the powers of a finite product.

Here we address the problem of establishing the finiteness properties of pairs of 2×2 sign-matrices. Such problem is related to the conjecture that pairs of sign-matrices fulfil the finiteness property for any dimension. This would imply, by a recent result by Jüngers and Blondel, that finite sets of rational matrices fulfil the finiteness property, which would be very important property in terms of the computation of the *joint spectral radius*. The finiteness property of $n \times n$ sign-matrices remains open but the knowledge of its validity in the 2×2-case gives the hope to solve it by using some induction argument on the dimension.

As a main tool of our proof we make use of a procedure to compute a so-called *real polytope extremal norm* for the set. In particular, we present an algorithm which, under some suitable assumptions, is able to check if a certain product in the multiplicative semigroup is spectrum maximizing.

For pairs of sign-matrices we develop the computations exactly and hence are able to prove analytically the finiteness property. On the other hand, the algorithm can be used in a floating point arithmetic and provide a general tool for approximating the joint spectral radius of a set of matrices.

Key words: Joint spectral radius, extremal norm, real polytope norm, finiteness property, sign matrices.

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1 Introduction

For a family $\mathscr{F} = \{A^{(i)}\}_{i \in \mathscr{I}}$, the following definitions are given in the literature. Let $\|\cdot\|$ be a given norm on the vector space $\mathbb{R}^n$ (or $\mathbb{C}^n$) and let the same symbol $\|\cdot\|$ also denote the corresponding induced $n \times n$-matrix norm. Then, for each $k = 0, 1, \ldots$, consider the set $\Sigma_k(\mathscr{F})$, with the convention that $\Sigma_0(\mathscr{F}) = \{I\}$. In the sequel, we shall make use of the notation

$$\Sigma(\mathscr{F}) = \bigcup_{k \geq 0} \Sigma_k(\mathscr{F}) \tag{1}$$

in order to indicate the multiplicative semigroup.

For each $k \geq 0$ set

$$\hat{\rho}_k(\mathscr{F}) = \sup_{P \in \Sigma_k(\mathscr{F})} \|P\| \tag{2}$$

and define the *joint spectral radius* of $\mathscr{F}$ as

$$\hat{\rho}(\mathscr{F}) = \limsup_{k \to \infty} \hat{\rho}_k(\mathscr{F})^{1/k}$$

(see [19]). Note that the numbers $\hat{\rho}_k(\mathscr{F})$ depend on the particular norm $\|\cdot\|$ used in (2) whereas, by the equivalence of all the norms in finite dimensional spaces, $\hat{\rho}(\mathscr{F})$ is independent of it.

Analogously, let $\rho(\cdot)$ denote the spectral radius of an $n \times n$-matrix and then, for each $k = 0, 1, \ldots$, consider

$$\bar{\rho}_k(\mathscr{F}) = \sup_{P \in \Sigma_k(\mathscr{F})} \rho(P)$$

and define the *generalized spectral radius* of $\mathscr{F}$ (see [5]) as

$$\bar{\rho}(\mathscr{F}) = \limsup_{k \to \infty} \bar{\rho}_k(\mathscr{F})^{1/k}.$$

Recently it has been shown that

$$\hat{\rho}(\mathscr{F}) = \bar{\rho}(\mathscr{F})$$

(see [2], [6], [21] and [20]). This means that the joint and the generalized spectral radius of $\mathscr{F}$ are the same number, which we shall simply call the *spectral radius* of the family of matrices $\mathscr{F}$ and denote by $\rho(\mathscr{F})$. Such result generalizes the well-known Gelfand theorem for a *single* matrix.

We introduce now a further characterization of the joint spectral radius. Given a norm $\|\cdot\|$ on the vector space $\mathbb{C}^n$ and the corresponding induced $n \times n$-matrix norm, we shall still use the same notation to define

$$\|\mathscr{F}\| = \hat{\rho}_1(\mathscr{F}) = \sup_{i \in \mathscr{I}} \|A^{(i)}\|.$$

The following result can be found, for example, in [19] and [6].

Theorem 1.1 *The spectral radius of a bounded family $\mathscr{F}$ of complex $n \times n$-matrices is characterized by the equality*

$$\rho(\mathscr{F}) = \inf_{\|\cdot\| \in \mathscr{N}} \|\mathscr{F}\|, \tag{3}$$

where $\mathscr{N}$ denotes the set of all possible induced $n \times n$-matrix norms.

Given a family $\mathscr{F}$, an important question to answer is whether or not the inf in (3) is actually attained by some induced matrix norm. To this purpose, we give the following definition.

Definition 1.1 *We shall say that a norm $\|\cdot\|_*$ satisfying the condition*

$$\|\mathscr{F}\|_* = \rho(\mathscr{F})$$

is extremal *for the family $\mathscr{F}$. A family of matrices which admits an extremal norm is said to be* non-defective *(see, for example, [9]).*

The actual computation of $\rho(\mathscr{F})$ is an important problem in several applications (see e.g. [10,11,16,17,1]) The problem, however, appears quite difficult in general (see e.g. [22]). Based on the inequalities

$$\bar{\rho}_k(\mathscr{F}) \le \rho(\mathscr{F})^k \le \hat{\rho}_k(\mathscr{F}) \quad \text{for all } k \ge 1,$$

(see [5]), an algorithm for efficiently computing lower bounds and upper bounds to $\rho(\mathscr{F})$ is proposed in [7]. Lately, further approaches for the approximation of the joint spectral radius have been considered (see for example [3,4] and [18]).

A way to compute exactly the joint spectral radius is based on the following property. If $\alpha > 0$ then

$$\rho(\mathscr{F}) = \alpha\rho\left(\frac{1}{\alpha}\mathscr{F}\right).$$

So, if $P \in \Sigma_k(\mathscr{F})$ is a certain product and

$$\alpha = \rho(P)^{1/k},$$

then we have that $\rho\left(\frac{1}{\alpha}\mathscr{F}\right) \ge 1$. Now, if we are able to find a norm such that $\|\frac{1}{\alpha}\mathscr{F}\| = 1$ we would have that

$$\alpha \le \rho(\mathscr{F}) \le \alpha \quad \Longrightarrow \quad \rho(\mathscr{F}) = \alpha = \rho(P)^{1/k}.$$

That would mean that the finiteness property holds and the product P determines the maximal growth ratio within $\Sigma(\mathscr{F})$ (see the forthcoming Definition 2.1).

The summary of the paper is the following. First, in Section 2, we introduce the main ideas of a procedure able to construct an extremal norm, which is obtained by applying the product semigroup to a suitable initial vector. Then we introduce real polytope norms, the main tool we use for the computation of the joint spectral radius, and present the procedure for the construction of an extremal norm. Then, in Section 3 we prove - case by case - the finiteness property for pairs of 2×2 sign-matrices. Finally, in Section 4 we outline some conclusions.

2 Constructing extremal real polytope norms

In this section we are concerned with the construction of an extremal norm for a non-defective family. To this aim, we need to review some basic definitions and results from the literature.

We recall that, if $\mathscr{X}$ is a subset of $\mathbb{R}^n$, then $\mathrm{co}(\mathscr{X},-\mathscr{X})$ is the set of all the finite convex linear combinations of vectors of $\{\mathscr{X},-\mathscr{X}\}$, i.e., $x\in\mathrm{co}(\mathscr{X},-\mathscr{X})$ if and only if there exist $x_1,\ldots,x_k\in\mathscr{X}$ with $k\geq 1$ such that

$$x=\sum_{i=1}^{k}\lambda_i x_i+\mu_i(-x_i)\quad\text{with}\quad \lambda_i,\mu_i\geq 0\quad\text{and}\quad \sum_{i=1}^{k}(\lambda_i+\mu_i)\leq 1.$$

Following [12], where the more general complex case was treated, we say that a bounded set $\mathscr{P}\subset\mathbb{R}^n$ is a *balanced real polytope* (b.r.p.) if there exists a finite set of vectors $\mathscr{X}=\{x_i\}_{1\leq i\leq m}$ such that $\mathrm{span}(\mathscr{X})=\mathbb{R}^n$ and

$$\mathscr{P}=\mathrm{co}(\mathscr{X},-\mathscr{X}). \tag{4}$$

Moreover, if $\mathrm{co}(\mathscr{X}',-\mathscr{X}')\subsetneq\mathrm{co}(\mathscr{X},-\mathscr{X})$ for all $\mathscr{X}'\subsetneq\mathscr{X}$, then the set $\mathscr{X}$ is called an *essential system of vertices* for $\mathscr{P}$, whereas any vector $\pm x_i$ is called a *vertex* of $\mathscr{P}$.

Clearly, the set $\mathscr{P}$ is the unit ball of a norm $\|\cdot\|_{\mathscr{P}}$ on $\mathbb{R}^n$, which we call a *real polytope norm* and is characterized as follows.

Lemma 2.1 *Let $\{\mathscr{X}\}_{i=1}^{m}$ be a set of vectors spanning $\mathbb{R}^n$ and $\mathscr{P}=\mathrm{co}(\mathscr{X},-\mathscr{X})$. Denote by $\|\cdot\|_{\mathscr{P}}$ the corresponding real polytope norm. Then, for any $z\in\mathbb{R}^n$, it holds that*

$$\|z\|_{\mathscr{P}}=\min_{\lambda_i\geq 0,\mu_i\geq 0}\left\{\sum_{i=1}^{m}(\lambda_i+\mu_i)\ \Big|\ z=\sum_{i=1}^{m}\lambda_i x_i+\mu_i(-x_i)\right\}. \tag{5}$$

It is well-known that real polytope norms are dense in the set of real vector norms. Similarly, the corresponding matrix norms are dense in the set of real induced operator norms.

Due to such density property we have the following result.

Theorem 2.1 *The spectral radius of a bounded family $\mathscr{F}$ of real $n \times n$-matrices is characterized by the equality $\rho(\mathscr{F}) = \inf_{\|\cdot\| \in \mathscr{N}_{pol}} \|\mathscr{F}\|$, where $\mathscr{N}_{pol}$ denotes the set of all possible induced $n \times n$-matrix real polytope norms.*

For some $P \in \Sigma_k(\mathscr{F})$ such that $\alpha = \rho(P)^{1/k} > 0$, it is convenient to consider a scaling of the original family $\mathscr{F} = \{A^{(i)}\}_{i \in \mathscr{I}}$ by the scalar α so as to obtain

$$\mathscr{F}^* = \left\{ \alpha^{-1} A^{(i)} \right\}_{i \in \mathscr{I}}.$$

In such a way we automatically have $\rho(\mathscr{F}^*) \geq 1$, an assumption which will be useful in the forthcoming Theorem 2.2.

Let us consider a (scaled) family $\mathscr{F}^*$ with $\rho(\mathscr{F}^*) \geq 1$. Then, for any vector $x \in \mathbb{R}^n$, we define the set (see (1))

$$\mathscr{T}[\mathscr{F}^*, x] = \{Px \mid P \in \Sigma(\mathscr{F}^*)\}, \tag{6}$$

i.e., the *trajectory* obtained by applying all the products P of matrices of $\mathscr{F}^*$ to the vector x.

The following theorem illustrates the possible use of the trajectory in the determination of an extremal norm.

Theorem 2.2 *Let $\mathscr{F}^*$ be a family of real $n \times n$-matrices such that $\rho(\mathscr{F}^*) \geq 1$ and, for a given $x \in \mathbb{R}^n$, let the trajectory $\mathscr{T}[\mathscr{F}^*, x]$ be a bounded subset of $\mathbb{R}^n$ such that* $\operatorname{span}\Big(\mathscr{T}[\mathscr{F}^*, x]\Big) = \mathbb{R}^n$. *Then we have that $\mathscr{F}^*$ is non-defective and $\rho(\mathscr{F}^*) = 1$. Furthermore,*

$$\mathscr{S}[\mathscr{F}^*, x] = \overline{\operatorname{co}\left(\mathscr{T}[\mathscr{F}^*, x], -\mathscr{T}[\mathscr{F}^*, x]\right)} \tag{7}$$

is the unit ball of an extremal norm $\|\cdot\|$ for $\mathscr{F}^*$ (that is, $\|\mathscr{F}^*\| = 1$).

The proof is analogous to that given in [13] in the more general context of complex families.

When $\rho(\mathscr{F}^*) = 1$, building the trajectory provides a tool for the construction of an extremal norm and, hence, for the computation of the spectral radius.

Assume that the hypotheses of Theorem 2.2 hold. The possibility of actually determining an extremal polytope norm, if any, is based on the search for a *suitable initial vector x* to which it corresponds a trajectory such that the set $\mathscr{S}[\mathscr{F}^*, x]$ is a symmetric real polytope. Such a choice is suggested by the recent result in [8] and is related to the knowledge (or the guess) of a spectrum maximizing product (see the forthcoming Definition).

Definition 2.1 (s.m.p.) *If $\mathscr{F}$ is a bounded family of complex $n \times n$-matrices, any matrix $\bar{P} \in \Sigma_k(\mathscr{F})$ satisfying*

$$\rho(\mathscr{F}) = \bar{\rho}_k(\mathscr{F})^{1/k} = \rho(\bar{P})^{1/k}$$

for some $k \geq 1$ will be called a spectrum-maximizing product *(*s.m.p.*) for $\mathscr{F}$.*

An s.m.p. is said minimal *if it is not a power of another s.m.p. of $\mathscr{F}$. Any eigenvector $x \neq 0$ of $\bar{P}$ related to an eigenvalue λ with $|\lambda| = \rho(\bar{P})$ is said to be a* leading eigenvector *of $\mathscr{F}$.*

In [8] we have proved that under some suitable conditions, which we do not discuss here, if a finite family $\mathscr{F}^*$ such that $\rho(\mathscr{F}^*) = 1$ of real $n \times n$-matrices has an s.m.p. P having a unique a leading eigenvector x, then it admits an extremal polytope norm. More specifically the set (see (6), (7))

$$\partial \mathscr{S}[\mathscr{F}^*, x] \bigcap \mathscr{T}[\mathscr{F}^*, x]$$

is finite. Hence there exists a finite number of products $\{P_k^*\}_{k=1}^{s} \in \Sigma(\mathscr{F}^*)$ such that

$$\mathscr{S}[\mathscr{F}^*, x] = \mathrm{co}\Big(\mathscr{X}, -\mathscr{X}\Big), \qquad \text{with } \mathscr{X} = \{P_k^* x\}_{k=1}^{s}.$$

We have also shown (see [8]) that, for a different choice of the initial vector, the finite convergence may not hold. Since we are interested in finiteness properties and hence in an exact determination of an extremal norm, such a choice of the initial vector turns out to be essential.

2.1 A procedure to construct an extremal real polytope norm

The following procedure is derived by a suitable development (restricted to the real case) of previous algorithms (see [14,13]).

We assume that $\mathscr{F}$ is finite, non-defective and irreducible (for the non-defective although reducible case we can proceed as in [13] and still make use of the procedure we propose). For a suitable initial vector x, the idea is that of computing iteratively the trajectory $\mathscr{T}[\mathscr{F}^*, x]$. While iterating, check whether $\mathscr{F}^*$ maps the convex hull of the balanced trajectory $\mathscr{T}[\mathscr{F}^*, x]$ into itself.

The idea of the following algorithm is that to apply recursively the scaled family $\mathscr{F}^*$ in order to construct the trajectory step by step starting from an initial vector.

Algorithm 2.1 *(for the construction of an extremal real polytope norm for $\mathscr{F} = \{A^{(1)}, \ldots, A^{(\ell)}\}$)*

```
Initialization

    Choose a candidate s.m.p. $P \in \Sigma_k(\mathscr{F})$ (for some $k$) and set $\alpha = \rho(P)^{1/k}$;
    define the scaled family $\mathscr{F}^* = \left\{\alpha^{-1} A^{(i)}\right\}_{i=1}^{\ell}$ (which is such that $\rho(\mathscr{F}^*) \geq 1$);
    compute a leading eigenvector $u$ of $P$ and set $v_0 = u$;
    define $\mathscr{V}^{(0)} = \mathscr{X}^{(0)} = \{v_0\}$, $J_1 = 1$ and set $\mathscr{P}^{(0)} = \mathrm{co}\left(\mathscr{V}^{(0)}, -\mathscr{V}^{(0)}\right)$;
    set $s = 1$.

Main iteration

(C) Let $\mathscr{X}^{(s-1)} = \{x_j\}_{j=1}^{J_s}$ and set $\mathscr{X}^{(s)} = \emptyset$;
    set $\mu_s = 1$ (the minimal admissible value for the norm to be constructed);
    for $i = 1, \ell$
        for $j = 1, J_s$
            compute the vector $y_j^i = A^{(i)} x_j$;
            compute the norm $\mu_j^i = \|y_j^i\|_{\mathscr{P}^{(s-1)}}$;
            if $\mu_j^i > 1$ and $\pm y_j^i \notin \mathscr{X}^{(s)}$ then insert $y_j^i$ into $\mathscr{X}^{(s)}$;
            if $\mu_j^i > \mu_s$ then set $\mu_s = \mu_j^i$;
        end
    end
    if $\mathscr{X}^{(s)} = \emptyset$ then stop (we have $\rho(\mathscr{F}^*) = 1$ and $\|\cdot\|_{\mathscr{P}^{(s-1)}}$ is an extremal norm);
    set $\mathscr{V}^{(s)} = \mathscr{V}^{(s-1)} \cup \mathscr{X}^{(s)}$;
    set $\mathscr{P}^{(s)} = \mathrm{co}\left(\mathscr{V}^{(s)}, -\mathscr{V}^{(s)}\right)$;
    compute an essential system of vertices $\mathscr{W}^{(s)} \subset \mathscr{V}^{(s)}$ of $\mathscr{P}^{(s)}$;
    update $\mathscr{V}^{(s)} = \mathscr{W}^{(s)}$;
    set $\mathscr{X}^{(s)} = \mathscr{X}^{(s)} \cap \mathscr{V}^{(s)}$ and $J_{s+1}$ = cardinality of $\mathscr{X}^{(s)}$;
    set $s = s + 1$ and go to (C).
```

If the procedure halts for some s, then, due to the irreducibility assumption, $\mathscr{P}^{(s-1)}$ determines the unit ball of an extremal real polytope norm for $\mathscr{F}^*$. In any case, the quantity $\mu_s = \|\mathscr{F}^*\|_{\mathscr{P}^{(s-1)}}$ provides an upper bound to $\rho(\mathscr{F}^*)$. Consequently,

$$\rho(\mathscr{F}) \leq \alpha \mu_s.$$

We remark that the initial choice of the product P may be obtained, for example, by means of the algorithm of Gripenberg [7], which progressively computes the quantities $\bar{\rho}_k(\mathscr{F})$ and $\hat{\rho}_k(\mathscr{F})$. Hence, a suitable choice for P is that of the product determining the lower bound $\bar{\rho}_k(\mathscr{F})$.

A stopping criterion

A useful criterion to stop the iteration and eventually discard the candidate s.m.p. P is given by the following theorem.

Theorem 2.3 *Let $\mathscr{F}$ be a finite irreducible family of matrices. If v_0 lies strictly inside $\mathscr{P}^{(s)}$, that is,*

$$v_0 \in \overset{\circ}{\mathscr{P}}{}^{(s)}, \tag{8}$$

at some step s of Algorithm 2.1, then $\rho(\mathscr{F}^*) > 1$. Viceversa, if $\rho(\mathscr{F}^*) > 1$, then there exists s such that (8) holds.

Proof. Assume that, at some step s, $v_0 \in \overset{\circ}{\mathscr{P}}{}^{(s)}$. This would mean that there exists $x_s \in \partial \mathscr{P}^{(s)}$ such that $x_s = \beta_s v_0$ with $\beta_s > 1$. Let $\mathscr{V}^{(s)} = \{v_i\}_{i=1}^m$ be such that $\{\mathscr{V}^{(s)}, -\mathscr{V}^{(s)}\}$ is an essential system of vertices of $\mathscr{P}^{(s)}$. This means that we can write

$$x_s = \sum_{i=1}^m \lambda_i v_i + \mu_i(-v_i) \qquad \text{with} \qquad \sum_{i=1}^m (\lambda_i + \mu_i) = 1.$$

Since, by construction, for all i there exists a finite product $P^{(i)} \in \Sigma(\mathscr{F}^*)$ such that $v_i = P^{(i)} v_0$, there must exist at least one product $P \in \Sigma(\mathscr{F}^*)$ such that

$$\|P v_0\|_{\mathscr{P}^{(s)}} = 1.$$

Since

$$1 = \|x_s\|_{\mathscr{P}^{(s)}} = \beta_s \|v_0\|_{\mathscr{P}^{(s)}},$$

we have

$$\|P v_0\|_{\mathscr{P}^{(s)}} = \beta_s \|v_0\|_{\mathscr{P}^{(s)}} > \|v_0\|_{\mathscr{P}^{(s)}} \implies \|P\|_{\mathscr{P}^{(s)}} \geq \beta_s > 1.$$

As a consequence $\|\mathscr{F}^*\|_{\mathscr{P}^{(s)}} > 1$.

Since $\mathscr{P}^{(s)} \subseteq \mathscr{P}^{(s+1)}$, we would still have $v_0 \in \overset{\circ}{\mathscr{P}}{}^{(s+1)}$ and the previous condition would occur for all subsequent values of s, with $\beta_{s+1} \geq \beta_s$.

If $\rho(\mathscr{F}^*) = 1$, by the irreducibility assumption,$\mathscr{P}^{(s)}$ would converge to some centrally symmetric convex set as $s \to \infty$.

As a consequence there would exist $\hat{s}$ such that $\|P\|_{\mathscr{P}^{(r)}} < \beta_s$ for all $r > \hat{s}$, which is not possible. Consequently $\rho(\mathscr{F}^*) > 1$.

Viceversa, by the irreducibility assumption, if $\rho(\mathscr{F}^*) > 1$ then

$$\lim_{s \to \infty} \mathscr{P}^{(s)} = \mathbb{R}^n.$$

This implies that there exists s such that $v_0 \in \overset{\circ}{\mathscr{P}}{}^{(s)}$. ■

Computation of the polytope norm

In order to compute the norm $\|z\|_{\mathscr{P}}$ where $\mathscr{P} = \mathrm{co}\,(\mathscr{X}, -\mathscr{X})$ with $\mathscr{X} = \{x_1, \ldots, x_m\}$, we observe that (5) is a standard linear programming problem of the form

$$\begin{aligned} \min \quad & f = \sum_{i=1}^{m} (\lambda_i + \mu_i) \\ \text{subject to} \quad & \sum_{i=1}^{m} \lambda_i x_i + \mu_i(-x_i) = z \\ \text{and} \quad & \lambda_i \geq 0, \mu_i \geq 0, \qquad i = 1, \ldots m, \end{aligned} \tag{9}$$

which can be solved efficiently.

Now we pass to consider the finiteness property of 2×2 sign matrices. We denote by $M_n(\mathbb{S})$ the set of $n \times n$ matrices with enties in $\{-1, 0, +1\}$. We recall the following conjecture by Blondel and Jüngers [15].

Conjecture 2.1 *Let $n \in \mathbb{N}$. Every pair of $n \times n$ sign-matrices fulfil the finiteness property.*

3 Finiteness properties of pairs of matrices in $M_2(\mathbb{S})$

We consider here the case of a family $\mathscr{F} = \{A, B\}$ where $A, B \in M_2(\mathbb{S})$.

The number of ordered pairs ($N_o = 3^4(3^4 - 1) = 6480$) is very large, but the number of cases to examine is immediately reduced to $N_e = N_o/8$, since the joint spectral radius of the sets $\{\pm A, \pm B\}$ does not change as well as it does not depend of the ordering of the two matrices. Hence $N_e = 810$, which is still a quite large number of cases. By using suitable properties, we shall see that the actual number of essential cases to examine is much lower.

Mainly, the properties we shall use are based on suitable similarity transformations, which do not change the joint spectral radius.

As in [15], in order to analyze the essential cases, we separate them into classes (n_0, n_1), where n_0 is the number of non-zero entries of A and n_1 is the number of non-zero entries of B. By simmetry, we can assume $n_0 \geq n_1$. Our approach consists in showing the finiteness property of every considered case by determining explicitly the associated s.m.p. through the construction of a suitable extremal norm. This does not allow a unified proof but, instead, requires to treat most of the essential cases separately.

Although all the pairs of binary matrices have already been considered in [15], here we reconsider the most difficult cases because our procedure is quite different from that used in [15] and does not rely on the possible non-negativity of the matrices.

The set of non-equivalent matrices with a single non-zero entry which has to be considered is given by $\mathbf{C} = \{C_i\}_{i=1}^{4}$ with

$$C_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad C_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The set of matrices with two non-zero entries which have to be considered is given by $\mathbf{D} = \{D_i\}_{i=1}^{11}$ with

$$D_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad D_3 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \quad D_4 = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix},$$

$$D_5 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad D_6 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad D_7 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \quad D_8 = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix},$$

$$D_9 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad D_{10} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad D_{11} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The set of matrices with three non-zero entries which have to be considered is given by $\mathbf{E} = \{E_i\}_{i=1}^{16}$ with

$$E_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix},$$

$$E_5 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad E_6 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad E_7 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \quad E_8 = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix},$$

$$E_9 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad E_{10} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad E_{11} = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}, \quad E_{12} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix},$$

$$E_{13} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad E_{14} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad E_{15} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \quad E_{16} = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}.$$

The set of matrices with four non-zero entries which have to be considered is given by $\mathbf{F} = \{F_i\}_{i=1}^{8}$ with

$$F_1 = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad F_3 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad F_4 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

$$F_5 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \quad F_6 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \quad F_7 = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad F_8 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Now we consider the similarity transformations associated to the matrices

$$P_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

which are such that $P_1^2 = I$, $P_2^2 = I$, $P_3^2 = -I$.

Clearly, we have that

$$P_k C_i P_k^{-1} \in \pm\mathbf{C}, \quad P_k D_i P_k^{-1} \in \pm\mathbf{D}, \quad P_k E_i P_k^{-1} \in \pm\mathbf{E}, \quad P_k F_i P_k^{-1} \in \pm\mathbf{F}. \tag{10}$$

In particular,

$$D_1 \sim D_2 \sim D_3 \sim D_4, \quad D_5 \sim D_6 \sim D_7 \sim D_8, \tag{11}$$

$$\begin{aligned} &E_1 \sim E_4 \sim E_{13} \sim E_{16}, \quad E_2 \sim E_3 \sim E_{14} \sim E_{15}, \\ &E_5 \sim E_6 \sim E_9 \sim E_{10}, \quad E_7 \sim E_8 \sim E_{11} \sim E_{12}, \end{aligned} \tag{12}$$

$$F_1 \sim F_4, \quad F_2 \sim F_3, \quad F_5 \sim F_6, \quad F_7 \sim F_8. \tag{13}$$

where $\sim$ denotes similarity.

In the sequel, we shall denote by $\mathscr{F}^* = \{A^*, B^*\}$ the scaled family to be proved to have joint spectral radius equal to 1.

In the following detailed analysis of all possible cases, some of them are treated in an easy way just by observing that one of the standard norms $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ is extremal.

Some other cases are easily treated by observing that the real polytope norm $\|\cdot\|_*^+$, associated to the b.r.p. $\mathscr{P}^+ = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2\}$, where

$$v_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

or $\|\cdot\|_*^-$, associated to the b.r.p. $\mathscr{P}^- = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2\}$, where

$$v_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

is extremal.

All the other cases are treated by using Algorithm 2.1.

The case $n_0 = 1$

The only possibility is $(n_0, n_1) = (1, 1)$, corresponding to families of the type $\mathscr{F} = \{C_i, C_j\}$ $(i < j)$.

The analysis is always trivial. In fact, it is very easy to see that $\rho(\mathscr{F}) = 1$ and any among $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ is an extremal norm. Moreover, if $i = 1$ or $j = 4$ an s.m.p. is $P = C_1$ or C_4, respectively. Only if $(i, j) = (1, 2)$ an s.m.p. is $P = C_2 C_3$.

The case $n_0 = 2$

In view of (10) and (11), we can restrict the choice of the first matrix A to the set $\mathbf{D}' = \{D_1, D_5, D_9, D_{10}, D_{11}\}$ and let the choice of B be free in $\mathbf{D}$.

The subcase $(n_0, n_1) = (2,1)$

It corresponds to families of the type $\mathscr{F} = \{D_i, C_j\}$.

Since $\|C_j\|_1 = \|C_j\|_\infty = 1$, $\rho(D_i) = 1$ and either $\|D_i\|_1 = 1$ or $\|D_i\|_\infty = 1$, we have that $\rho(\mathscr{F}) = 1$ and that an s.m.p. is $P = D_i$.

The subcase $(n_0, n_1) = (2,2)$

It corresponds to families of the type $\mathscr{F} = \{D_i, D_j\}$.

- $A = D_1$ and $B = D_j$ $(j = 2,3,4,9,10,11)$.
  Since $\rho(A) = \rho(B) = \|A\|_1 = \|B\|_1 = 1$, we have that $\rho(\mathscr{F}) = 1$ and that A and B are both s.m.p.s.
- $A = D_1$ and $B = D_5$.
  We find that $P = AB$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/2} = \sqrt{2}$ and an extremal polytope norm is given $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2\}$, where v_0 is the leading eigenvector of P, $v_1 = A^* v_0$, $v_2 = B^* v_0$.
- $A = D_1$ and $B = D_6$.
  We find that $P = B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P) = 1$ and an extremal polytope norm is given $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1\}$, where v_0 is the leading eigenvector of P, $v_1 = A^* v_0$.
- $A = D_1$ and $B = D_j$ $(j = 7,8)$.
  Since $A^2 = A$, $B^2 = B$, $\rho(AB) = \rho(BA) = 0$ and $\rho(A) = \rho(B) = 1$, we have that $\rho(\mathscr{F}) = 1$ and that A and B are both s.m.p.s.
- $A = D_5$ and $B \in \mathbf{D}$.
  Since $D_5 = D_1^{\mathrm{T}}$ and $\mathbf{D}^T \subseteq \pm\mathbf{D}$ and since, if P is an s.m.p. of the family $\mathscr{F} = \{A, B\}$, then P^{T} is an s.m.p. of the family $\mathscr{F}^T = \{A^{\mathrm{T}}, B^{\mathrm{T}}\}$, we are led again to the previous cases.
- $A = D_j$ and $B = D_k$ $(j = 9,10,11,\ k = 2,3,4,6,7,8)$.
  Since $P_1 D_9 P_1^{-1} = -D_9$, $P_2 D_9 P_2^{-1} = D_9$, $P_3 D_9 P_3^{-1} = -D_9$, $P_1 D_{10} P_1^{-1} = D_{10}$, $P_2 D_{10} P_2^{-1} = -D_{10}$, $P_3 D_{10} P_3^{-1} = -D_{10}$, $P_1 D_{11} P_1^{-1} = -D_{11}$, $P_2 D_{11} P_2^{-1} = -D_{11}$, $P_3 D_{11} P_3^{-1} = D_{11}$ and since $P_1 D_2 P_1^{-1} = D_1$, $P_2 D_3 P_2^{-1} = D_1$, $P_3 D_4 P_3^{-1} = -D_1$, $P_1 D_6 P_1^{-1} = D_5$, $P_2 D_7 P_2^{-1} = D_5$, $P_3 D_8 P_3^{-1} = -D_5$, by using the similarity transformations associated to P_1, P_2 and P_3 we are led to the previous cases.
- $A = D_j$ and $B = D_k$ $(j, k = 9,10,11)$.
  Since $\rho(A) = \rho(B) = \|A\|_\infty = \|B\|_\infty = 1$, we have that $\rho(\mathscr{F}) = 1$ and that A and B are both s.m.p.s.

The case $n_0 = 3$

In view of (10) and (12), we can restrict the choice of the first matrix A to the set $\mathbf{E}' = \{E_1, E_2, E_5, E_7\}$ and let the choice of B be free.

The subcase $(n_0, n_1) = (3, 1)$.

It corresponds to families of the type $\mathscr{F} = \{E_i, C_j\}$.

- $A = E_1$, $B \in \mathbf{C}$.
  We find that $P = A$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P) = \frac{1+\sqrt{5}}{2}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1\}$, where v_0 is the leading eigenvector of P, $v_1 = B^* v_0$.
- $A = E_5$, $B = C_j$ $(j = 1, 2, 4)$.
  The family $\mathscr{F}$ is upper triangular and defective with $\rho(\mathscr{F}) = 1$ and A is an s.m.p..
- $A = E_5$, $B = C_3$.
  We find that $P = A^4 B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P) = 4^{1/5}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5\}$, where v_0 is the leading eigenvector of P, $v_1 = A^* v_0$, $v_2 = B^* v_0$, $v_3 = A^* v_2$, $v_4 = A^* v_3$, $v_5 = A^* v_4$.
- $A = E_2$, $B \in \mathbf{C}$.
  Since $\rho(A) = \|A\|_*^+ = \|B\|_*^+ = 1$, we have that $\rho(\mathscr{F}) = 1$ and that A is an s.m.p..
- $A = E_7$, $B \in \mathbf{C}$.
  Since $\rho(A) = \|A\|_*^- = \|B\|_*^- = 1$, we have that $\rho(\mathscr{F}) = 1$ and that A is an s.m.p..

The subcase $(n_0, n_1) = (3, 2)$

It corresponds to families of the type $\mathscr{F} = \{E_i, D_j\}$.

- $A = E_1$, $B \in \mathbf{D}$.
  Since $\rho(A) = \|A\|_2 = \frac{1+\sqrt{5}}{2}$ and $\|B\|_2 \leq \sqrt{2}$, we have that $\rho(\mathscr{F}) = 1$ and that A is an s.m.p..
- $A = E_2$, $B = D_1$.
  We find that $P = AB$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/2} = \sqrt{2}$ and an extremal polytope norm is given $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2\}$, where v_0 is the leading eigenvector of P, $v_1 = B^* v_0$, $v_2 = A^* v_0$.
- $A = E_2$, $B = D_j$ $(j = 2, 8)$.
  We find that $P = A^2 B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/3} = 2^{1/3}$ and an extremal polytope norm is given $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2\}$, where v_0 is the leading eigenvector of P, $v_1 = B^* v_0$, $v_2 = A^* v_1$.
- $A = E_2$, $B = D_j$ $(j = 3, 4, 5, 6, 10)$.

Since $\rho(A) = \rho(B) = \|A\|_*^+ = \|B\|_*^+ = 1$, we have that $\rho(\mathscr{F}) = 1$ and that A and B are both s.m.p.s.

- $A = E_2, B = D_7$.
  We find that $P = AB$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/2} = \sqrt{2}$ and an extremal polytope norm is given $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2\}$, where v_0 is the leading eigenvector of P, $v_1 = B^* v_0$, $v_2 = A^* v_1$.
- $A = E_2, B = D_9$.
  We find that $P = AB$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/2} = \left(\frac{1+\sqrt{5}}{2}\right)^{1/2}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2\}$, where v_0 is the leading eigenvector of P, $v_1 = B^* v_0$, $v_2 = A^* v_0$.
- $A = E_2, B = D_{11}$.
  We find that $P = ABA^2B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/5} = \left(\frac{3+\sqrt{5}}{2}\right)^{1/5}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5\}$, where v_0 is the leading eigenvector of P, $v_1 = B^* v_0$, $v_2 = A^* v_1$, $v_3 = A^* v_2$, $v_4 = B^* v_2$, $v_5 = B^* v_3$.
- $A = E_5, B = D_j$ $(j = 1, 3, 6, 8, 9)$.
  The family $\mathscr{F}$ is upper triangular and defective with $\rho(\mathscr{F}) = 1$ and both A and B are s.m.p.s.
- $A = E_5, B = D_j$ $(j = 2, 5)$.
  We find that $P = A^2B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/3} = 3^{1/3}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4\}$, where v_0 is the leading eigenvector of P, $v_1 = A^* v_0$, $v_2 = B^* v_0$, $v_3 = A^* v_1$, $v_4 = A^* v_2$.
- $A = E_5, B = D_j$ $(j = 4, 7)$.
  We find that $P = A^5B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/6} = 2^{1/3}$ and an extremal polytope norm is given $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6\}$, where v_0 is the leading eigenvector of P, $v_1 = A^* v_0$, $v_2 = B^* v_0$, $v_3 = A^* v_2$, $v_4 = A^* v_3$, $v_5 = A^* v_4$, $v_6 = A^* v_5$.
- $A = E_5, B = D_{10}$.
  We find that $P = A^3B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/4} = \left(\frac{3+\sqrt{13}}{2}\right)^{1/4}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5\}$, where v_0 is the leading eigenvector of P, $v_1 = A^* v_0$, $v_2 = B^* v_0$, $v_3 = A^* v_2$, $v_4 = B^* v_1$, $v_5 = A^* v_3$.
- $A = E_5, B = D_{11}$.
  We find that $P = A^4B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/5} = (2 + \sqrt{3})^{1/5}$ and an extremal polytope norm is given $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6\}$, where v_0 is the leading eigenvector of P, $v_1 = A^* v_0$, $v_2 = B^* v_0$, $v_3 = A^* v_2$, $v_4 = A^* v_3$, $v_5 = A^* v_4$, $v_6 = B^* v_5$.
- $A = E_7, B = D_j$ $(j = 1, 3, 6, 8, 9)$.
  The family $\mathscr{F}$ is upper triangular with $\rho(\mathscr{F}) = 1$ and A and B are both s.m.p.s.
- $A = E_7, B = D_j$ $(j = 2, 7, 10)$.
  Since $\rho(A) = \rho(B) = \|A\|_*^- = \|B\|_*^- = 1$, we have that $\rho(\mathscr{F}) = 1$ and that A and B are both s.m.p.s.

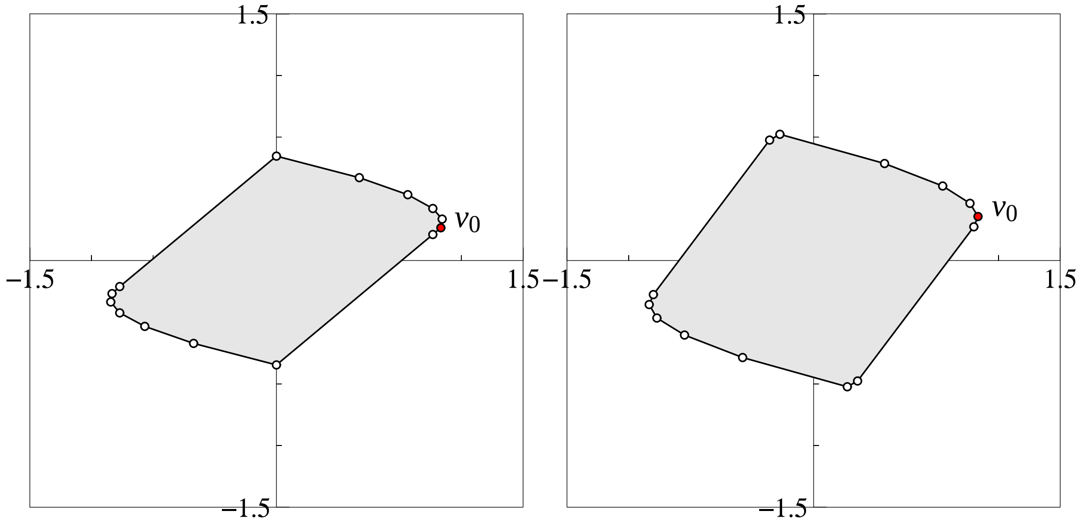

Fig. 1. Polytope norm for the pair $\{A=E_5, B=D_4\}$ (left) and for the pair $\{A=E_5, B=D_{11}\}$ (right).

- $A=E_7$, $B=D_j$ $(j=4,5)$.
  We find that $P=AB$ is an s.m.p., $\rho(\mathscr{F})=\rho(P)^{1/2}=\sqrt{2}$ and an extremal polytope norm is given by $\mathscr{P}=\mathrm{co}\,(V,-V)$ with $V=\{v_0,v_1\}$, where v_0 is the leading eigenvector of P and $v_1=B^*v_0$.
- $A=E_7$, $B=D_{11}$.
  We find that $P=AB$ is an s.m.p., $\rho(\mathscr{F})=\rho(P)^{1/2}=\left(\frac{1+\sqrt{5}}{2}\right)^{1/2}$ and an extremal polytope norm is given by $\mathscr{P}=\mathrm{co}\,(V,-V)$ with $V=\{v_0,v_1\}$, where v_0 is the leading eigenvector of P and $v_1=B^*v_0$.

Now we give a detailed proof of the case $A=E_2$, $B=D_{11}$.

Let $\gamma=\frac{1}{\rho(\mathscr{F})}$. Then we get

$$v_0=\begin{pmatrix}1\\ \frac{2}{1+\sqrt{5}}\end{pmatrix},\ v_1=\gamma\begin{pmatrix}\frac{2}{1+\sqrt{5}}\\ -1\end{pmatrix},\ v_2=\gamma^2\begin{pmatrix}\frac{3+\sqrt{5}}{1+\sqrt{5}}\\ \frac{2}{1+\sqrt{5}}\end{pmatrix},$$
$$v_3=\gamma^3\begin{pmatrix}1\\ \frac{3+\sqrt{5}}{1+\sqrt{5}}\end{pmatrix},\ v_4=\gamma^3\begin{pmatrix}\frac{2}{1+\sqrt{5}}\\ -\frac{3+\sqrt{5}}{1+\sqrt{5}}\end{pmatrix},\ v_5=\gamma^4\begin{pmatrix}\frac{3+\sqrt{5}}{1+\sqrt{5}}\\ -1\end{pmatrix}.$$

As illustrated in Figure 2, we analyze the transformed vectors $\mathscr{F}^*(V)$. Some of them are vertices themselves by construction of $\mathscr{P}$ and, hence, do not need to be analyzed. Here we report such vectors together with the minimizing convex combinations of vertices of $\mathscr{P}$ which determine their norms (see (9)):

$$A^*v_0=\gamma\begin{pmatrix}\frac{\sqrt{5}-1}{1+\sqrt{5}}\\ 1\end{pmatrix}=\lambda v_3+\mu(-v_4),\quad \lambda=\frac{2(3+\sqrt{5})}{\gamma^2(11+5\sqrt{5})},\ \mu=\frac{2}{\gamma^2(7+3\sqrt{5})},$$

$$\|A^*v_0\|_{\mathscr{P}}=\lambda+\mu\approx 0.90;$$

$$A^*v_1=v_2;$$

$$A^*v_2=v_3;$$

$$A^*v_3=\lambda(-v_1),\qquad \lambda=\gamma^3,\quad \|A^*v_3\|_{\mathscr{P}}=\lambda\approx 0.56;$$

$$A^*v_4=\gamma^4\begin{pmatrix}\frac{5+\sqrt{5}}{1+\sqrt{5}}\\ \frac{2}{1+\sqrt{5}}\end{pmatrix}=\lambda v_2+\mu v_5,\quad \lambda=\frac{4(2+\sqrt{5})}{7+3\sqrt{5}}\gamma^2,\ \mu=\frac{2}{7+3\sqrt{5}},$$

$$\|A^*v_4\|_{\mathscr{P}}=\lambda+\mu\approx 0.98;$$

$$A^*v_5=v_0;$$

$$B^*v_0=v_1;$$

$$B^*v_1=\lambda(-v_0),\qquad \lambda=\gamma^2,\quad \|B^*v_0\|_{\mathscr{P}}=\lambda\approx 0.68;$$

$$B^*v_2=v_4;$$

$$B^*v_3=v_5;$$

$$B^*v_4=\lambda(-v_2),\qquad \lambda=\gamma^2,\quad \|B^*v_4\|_{\mathscr{P}}=\lambda\approx 0.68;$$

$$B^*v_5=\lambda v_3,\qquad \lambda=\gamma^2,\quad \|B^*v_5\|_{\mathscr{P}}=\lambda\approx 0.68.$$

This proves the extremality of $\|\cdot\|_{\mathscr{P}}$ and that $P=ABA^2B$ is an s.m.p..

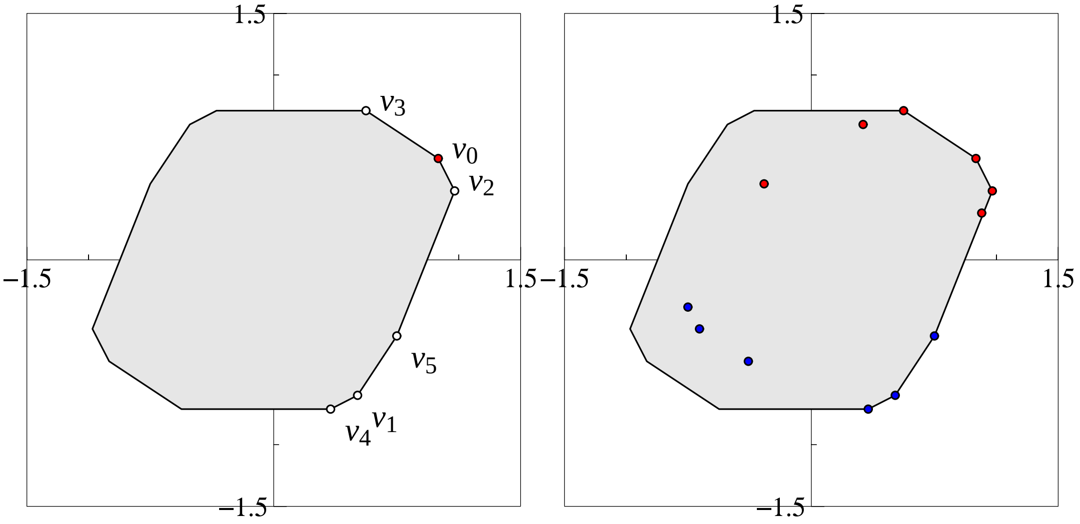

Fig. 2. Polytope norm for the pair $\{A=E_2,\,B=D_{11}\}$ (left) and the set $\mathscr{F}^*(V)$ (right). Red points indicate the vectors $\{A^*v_i\}_{i=0}^5$ and blue points indicate the vectors $\{B^*v_i\}_{i=0}^5$.

The subcase $(n_0,n_1)=(3,3)$.

It corresponds to families of the type $\mathscr{F}=\{E_i,E_j\}$.

- $A=E_1$, $B\in\mathbf{E}$.
  Since $\rho(A)=\|A\|_2=\|B\|_2=\frac{1+\sqrt{5}}{2}$, we have that $\rho(\mathscr{F})=\frac{1+\sqrt{5}}{2}$ and that A is an s.m.p..
- $A=E_2$, $B=E_3$.
  We find that $P=AB$ is an s.m.p., $\rho(\mathscr{F})=\rho(P)^{1/2}=\frac{1+\sqrt{5}}{2}$ and an extremal polytope norm is given by $\mathscr{P}=\mathrm{co}\,(V,-V)$ with $V=\{v_0,v_1\}$, where v_0 is the leading eigenvector of P and $v_1=B^*v_0$.
- $A=E_2$, $B=E_j$ $(j=4,13,16)$. Since $\rho(B)=\|A\|_2=\|B\|_2=\frac{1+\sqrt{5}}{2}$, we have that $\rho(\mathscr{F})=\frac{1+\sqrt{5}}{2}$ and that B is an s.m.p..
- $A=E_2$, $B=E_j$ $(j=5,10)$.
  We find that $P=AB^3$ is an s.m.p., $\rho(\mathscr{F})=\rho(P)^{1/4}=(2+\sqrt{3})^{1/4}$ and an extremal polytope norm is given by $\mathscr{P}=\mathrm{co}\,(V,-V)$ with $V=\{v_0,v_1,v_2,v_3,v_4,v_5,v_6,v_7\}$, where v_0 is the leading eigenvector of P, $v_1=A^*v_0$, $v_2=B^*v_0$, $v_3=A^*v_2$, $v_4=B^*v_2$, $v_5=A^*v_4$, $v_6=B^*v_4$, $v_7=B^*v_6$.
- $A=E_2$, $B=E_j$ $(j=6,9)$.
  We find that $P=A^2B^3$ is an s.m.p., $\rho(\mathscr{F})=\rho(P)^{1/5}=(2+\sqrt{3})^{1/5}$ and an extremal polytope norm is given by $\mathscr{P}=\mathrm{co}\,(V,-V)$ with $V=\{v_0,v_1,v_2,v_3,v_4,v_5,v_6\}$, where v_0 is the leading eigenvector of P, $v_1=B^*v_0$, $v_2=B^*v_1$, $v_3=B^*v_2$, $v_4=A^*v_3$, $v_5=B^*v_3$, $v_6=A^*v_5$.
- $A=E_2$, $B=E_j$ $(j=7,12)$.
  We find that $P=AB$ is an s.m.p., $\rho(\mathscr{F})=\rho(P)^{1/2}=(1+\sqrt{2})^{1/2}$ and an extremal polytope norm is given by $\mathscr{P}=\mathrm{co}\,(V,-V)$ with $V=\{v_0,v_1,v_2\}$, where v_0 is the leading eigenvector of P, $v_1=A^*v_0$, $v_2=B^*v_0$.
- $A=E_2$, $B=E_j$ $(j=8,11,15)$.
  Since $\rho(A)=\rho(B)=\|A\|_*^+=\|B\|_*^+=1$, we have that $\rho(\mathscr{F})=1$ and that A and B are both s.m.p.s.
- $A=E_2$, $B=E_{14}$.
  We find that $P=ABA^2BAB^2$ is an s.m.p., $\rho(\mathscr{F})=\rho(P)^{1/8}=(7+4\sqrt{3})^{1/8}$ and an extremal polytope norm is given by $\mathscr{P}=\mathrm{co}\,(V,-V)$ with $V=\{v_0,v_1,v_2,v_3,v_4,v_5,v_6,v_7\}$, where v_0 is the leading eigenvector of P, $v_1=B^*v_0$, $v_2=B^*v_1$, $v_3=A^*v_2$, $v_4=B^*v_3$, $v_5=A^*v_4$, $v_6=A^*v_5$, $v_7=B^*v_6$.

  Observe that this is the first of the two cases with the largest number of factors in the s.m.p.. The essential vertices of $\mathscr{P}$ are just the leading eigenvectors of $\mathscr{F}$, that is, the eigenvectors of all the cyclic permutations of P.
- $A=E_5$, $B=E_j$ $(j=3,15)$.
  We find that $P=A^3B^2$ is an s.m.p., $\rho(\mathscr{F})=\rho(P)^{1/5}=\left(2+\sqrt{3}\right)^{1/5}$ and an extremal polytope norm is given by $\mathscr{P}=\mathrm{co}\,(V,-V)$ with $V=\{v_0,v_1,v_2,v_3,v_4,v_5,v_6\}$, where v_0 is the leading eigenvector of P, $v_1=A^*v_0$, $v_2=B^*v_0$, $v_3=B^*v_1$, $v_4=B^*v_2$, $v_5=A^*v_4$, $v_6=A^*v_5$.

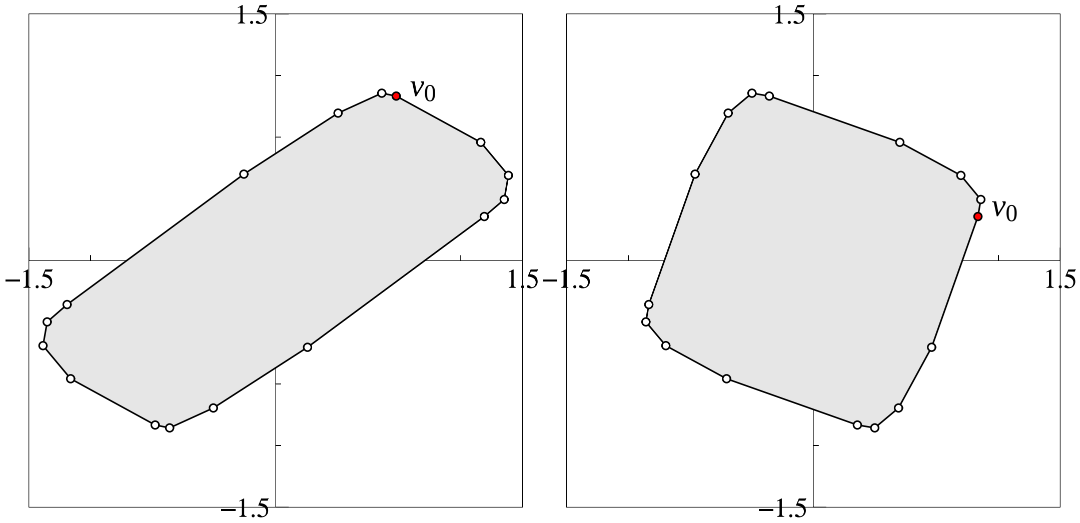

Fig. 3. Polytope norm for the pair $\{A = E_2, B = E_5\}$ (left) and for the pair $\{A = E_2, B = E_{10}\}$ (right).

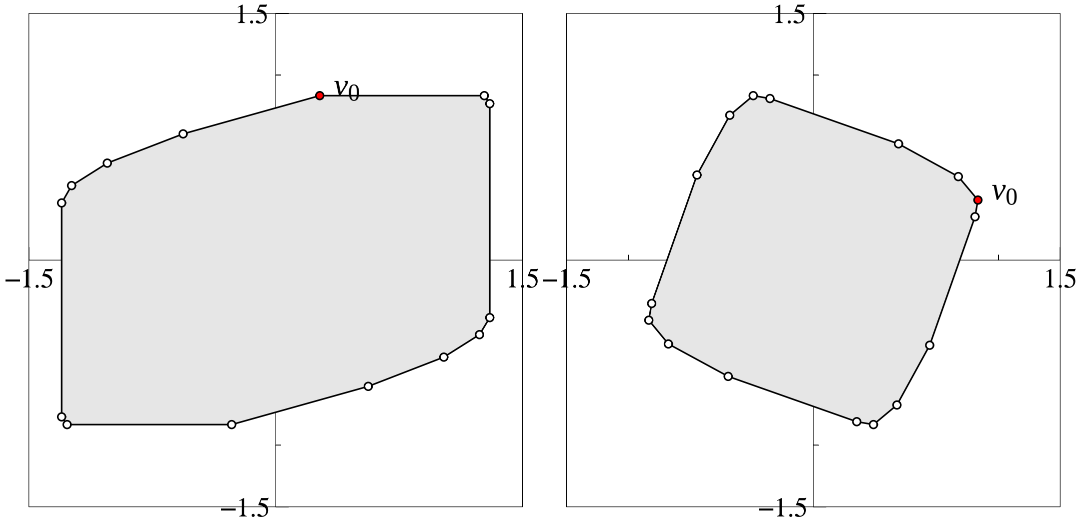

Fig. 4. Polytope norm for the pair $\{A = E_2, B = E_6\}$ (left) and for the pair $\{A = E_2, B = E_{14}\}$ (right).

- $A = E_5$, $B = E_j$ $(j = 4, 13, 16)$.
  Since $\rho(B) = \|A\|_2 = \|B\|_2 = \frac{1+\sqrt{5}}{2}$, we have that $\rho(\mathscr{F}) = \frac{1+\sqrt{5}}{2}$ and that B is an s.m.p..
- $A = E_5$, $B = E_j$ $(j = 6, 7, 8)$.
  The family $\mathscr{F}$ is upper triangular and defective with $\rho(\mathscr{F}) = 1$ and both A and B are s.m.p.s..
- $A = E_5$, $B = E_9$.
  We find that $P = AB$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/2} = \frac{1+\sqrt{5}}{2}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4\}$, where v_0 is the leading eigenvector of P and $v_1 = A^* v_0$, $v_2 = B^* v_0$, $v_3 = A^* v_1$, $v_4 = B^* v_2$.

- $A = E_5$, $B = E_{10}$.
  We find that $P = A^4B^4$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/8} = \left(7+4\sqrt{3}\right)^{1/8}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$, where v_0 is the leading eigenvector of P, $v_1 = B^*v_0$, $v_2 = B^*v_1$, $v_3 = B^*v_2$, $v_4 = B^*v_3$, $v_5 = A^*v_4$, $v_6 = A^*v_5$, $v_7 = A^*v_6$.
  This is the second of the two cases with the largest number of factors in the s.m.p.. Again, the essential vertices of $\mathscr{P}$ are just the leading eigenvectors of $\mathscr{F}$.
- $A = E_5$, $B = E_j$ ($j = 11, 12$).
  We find that $P = A^3B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/4} = \left(\frac{3+\sqrt{13}}{2}\right)^{1/4}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5\}$, where v_0 is the leading eigenvector of P, $v_1 = A^*v_0$, $v_2 = B^*v_0$, $v_3 = B^*v_1$, $v_4 = A^*v_2$, $v_5 = A^*v_4$.
- $A = E_5$, $B = E_{14}$.
  We find that $P = A^3B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/4} = \left(2+\sqrt{3}\right)^{1/4}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$, where v_0 is the leading eigenvector of P, $v_1 = A^*v_0$, $v_2 = B^*v_0$, $v_3 = A^*v_2$, $v_4 = B^*v_2$, $v_5 = A^*v_3$, $v_6 = B^*v_3$, $v_7 = B^*v_5$.

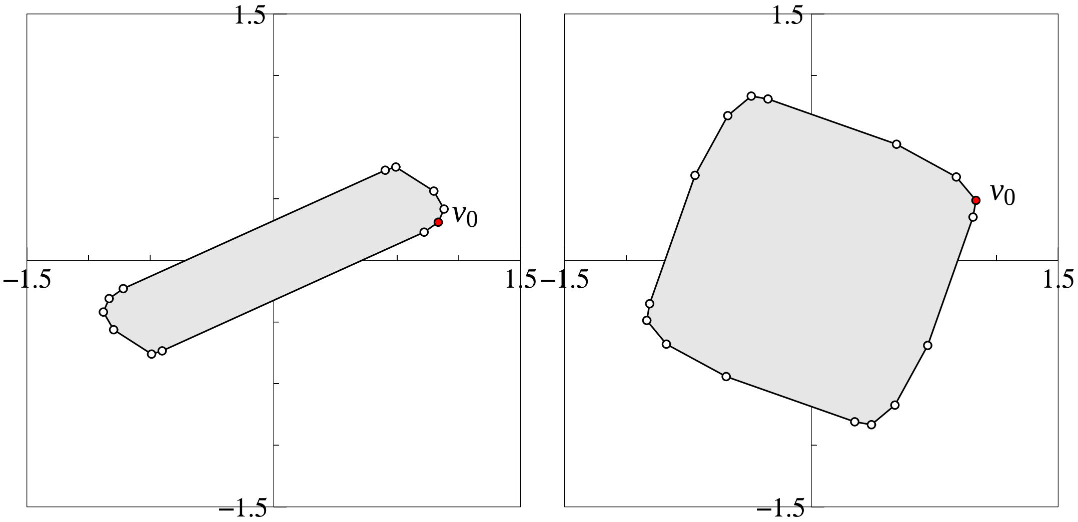

Fig. 5. Polytope norm for the pair $\{A = E_5, B = E_{11}\}$ (left) and for the pair $\{A = E_5, B = E_{14}\}$ (right).

- $A = E_7$, $B = E_j$ ($j = 3, 12, 14$).
  Since $\rho(A) = \rho(B) = \|A\|_*^- = \|B\|_*^- = 1$, we have that $\rho(\mathscr{F}) = 1$ and that A and B are both s.m.p.s.
- $A = E_7$, $B = E_j$ ($j = 4, 13, 16$).
  Since $\rho(B) = \|A\|_2 = \|B\|_2 = \frac{1+\sqrt{5}}{2}$, we have that $\rho(\mathscr{F}) = \frac{1+\sqrt{5}}{2}$ and that B is an s.m.p..
- $A = E_7$, $B = E_j$ ($j = 6, 8$).
  The family $\mathscr{F}$ is upper triangular and defective with $\rho(\mathscr{F}) = 1$ and both A and B are s.m.p.s..

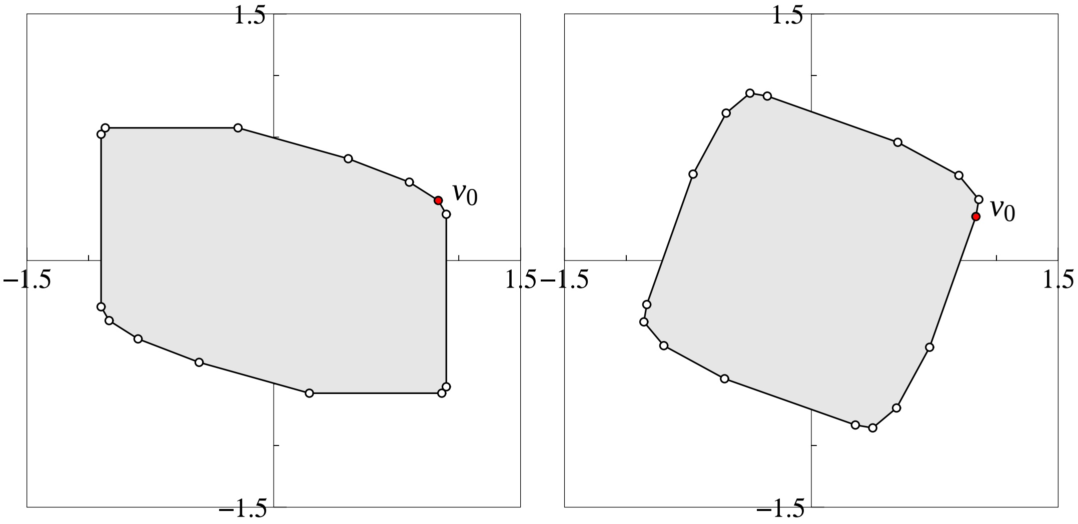

Fig. 6. Polytope norm for the pair $\{A=E_5,B=E_3\}$ (left) and for the pair $\{A=E_5,B=E_{10}\}$ (right).

- $A=E_7$, $B=E_j$ $(j=9,10)$.
  We find that $P=AB^3$ is an s.m.p., $\rho(\mathscr{F})=\rho(P)^{1/5}=\left(\frac{3+\sqrt{13}}{2}\right)^{1/4}$ and an extremal polytope norm is given by $\mathscr{P}=\mathrm{co}\,(V,-V)$ with $V=\{v_0,v_1,v_2,v_3,v_4,v_5,v_6\}$, where v_0 is the leading eigenvector of P, $v_1=B^*v_0$, $v_2=B^*v_1$, $v_3=B^*v_2$, $v_4=B^*v_3$, $v_5=A^*v_4$.
- $A=E_7$, $B=E_{11}$.
  We find that $P=AB$ is an s.m.p., $\rho(\mathscr{F})=\rho(P)^{1/2}=\frac{1+\sqrt{5}}{2}$ and an extremal polytope norm is given by $\mathscr{P}=\mathrm{co}\,(V,-V)$ with $V=\{v_0,v_1\}$, where v_0 is the leading eigenvector of P and $v_1=B^*v_0$.
- $A=E_7$, $B=E_{15}$.
  We find that $P=AB$ is an s.m.p., $\rho(\mathscr{F})=\rho(P)^{1/2}=\left(1+\sqrt{2}\right)^{1/2}$ and an extremal polytope norm is given by $\mathscr{P}=\mathrm{co}\,(V,-V)$ with $V=\{v_0,v_1,v_2\}$, where v_0 is the leading eigenvector of P, $v_1=B^*v_0$, $v_2=B^*v_1$.

The case $n_0=4$

In view of (10) and (13), we can restrict the choice of the first matrix A to the set $\mathbf{F}'=\{F_1,F_3,F_5,F_8\}$ and let the choice of B be free.

The subcase $(n_0,n_1)=(4,1)$.

It corresponds to families of the type $\mathscr{F}=\{F_i,C_j\}$.

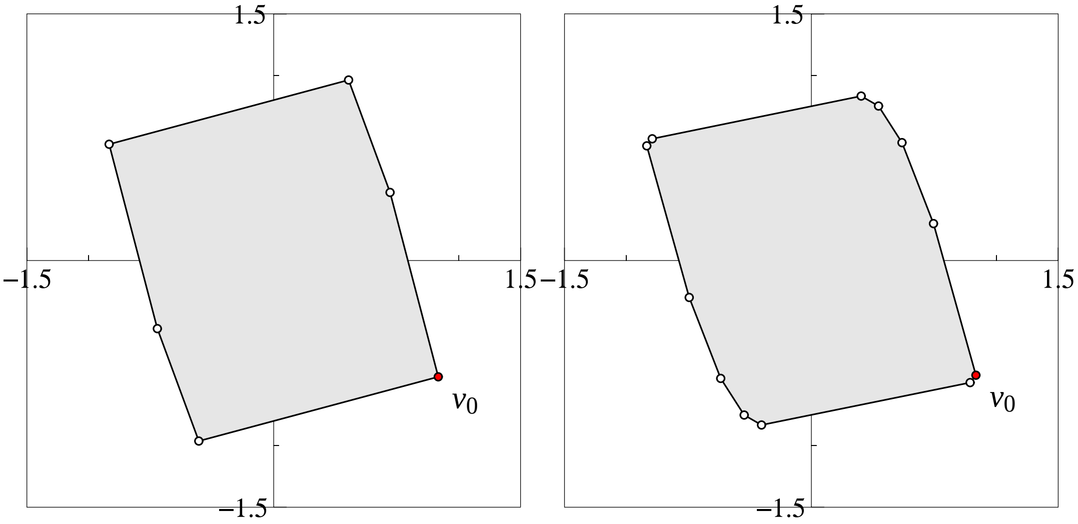

Fig. 7. Polytope norm for the pair $\{A = E_7, B = E_{15}\}$ (left) and for the pair $\{A = E_7, B = E_9\}$ (right).

- $A = F_i$ $(i = 1,3)$, $B \in \mathbf{C}$.
  Since $\rho(A) = \|A\|_2 = \sqrt{2}$ and $\|B\|_2 = 1$, we have that $\rho(\mathscr{F}) = \sqrt{2}$ and that A is an s.m.p..
- $A = F_5$, $B = C_j$ $(j = 1,4)$.
  Since $\rho(B) = \|A\|_*^- = \|B\|_*^- = 1$, we have that $\rho(\mathscr{F}) = 1$ and that B is an s.m.p..
- $A = F_5$, $B = C_j$ $(j = 2,3)$.
  Since $\rho(AB) = \|A\|_*^- = \|B\|_*^- = 1$, we have that $\rho(\mathscr{F}) = 1$ and that $P = AB$ is an s.m.p..
- $A = F_8$, $B \in \mathbf{C}$.
  Since $\rho(A) = \|A\|_1 = 2$ and $\|B\|_1 = 1$, we have that $\rho(\mathscr{F}) = 2$ and that A is an s.m.p..

The subcase $(n_0, n_1) = (4,2)$.

It corresponds to families of the type $\mathscr{F} = \{F_i, D_j\}$.

- $A = F_i$ $(i = 1,3)$, $B \in \mathbf{D}$.
  Since $\rho(A) = \|A\|_2 = \sqrt{2}$ and $\|B\|_2 \leq \sqrt{2}$, we have that $\rho(\mathscr{F}) = \sqrt{2}$ and that A is an s.m.p..
- $A = F_5$, $B = D_j$ $(j = 1,2,7,8,10)$.
  Since $\rho(B) = \|A\|_*^- = \|B\|_*^- = 1$, we have that $\rho(\mathscr{F}) = 1$ and that B is an s.m.p..
- $A = F_5$, $B = D_j$ $(j = 3,4,5,6,9,11)$.
  We find that $P = AB$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/2} = \sqrt{2}$ and an extremal polytope norm is given by $\mathscr{P} = \operatorname{co}(V, -V)$ with $V = \{v_0, v_1\}$, where v_0 is the leading eigenvector of P and $v_1 = B^* v_0$.
- $A = F_8$, $B \in \mathbf{D}$.

Since $\rho(A) = \|A\|_2 = 2$ and $\|B\|_2 \le \sqrt{2}$, we have that $\rho(\mathscr{F}) = 2$ and that A is an s.m.p..

The subcase $(n_0, n_1) = (4, 3)$.

It corresponds to families of the type $\mathscr{F} = \{F_i, E_j\}$.

It is useful to observe that $P_3 F_1 P_3^{-1} = -F_1$, $P_3 F_3 P_3^{-1} = F_3$, $P_1 F_5 P_1^{-1} = -F_5$ and that both the similarity transformations associated to P_1 and P_3 are one-to-one applications between the sets of matrices $\mathbf{E}'' = \{E_j \mid 1 \le j \le 8\}$ and $\mathbf{E}''' = \{E_j \mid 9 \le j \le 16\}$. Consequently, when $A = F_i$ $(i = 1, 3, 5)$, we can restrict the choice of the matrix B within the set $\mathbf{E}''$.

- $A = F_1$, $B = E_j$ $(j = 1, 4)$.
  Since $\rho(B) = \|B\|_2 = \frac{1+\sqrt{5}}{2}$ and $\|A\|_2 = \sqrt{2}$, we have that $\rho(\mathscr{F}) = \frac{1+\sqrt{5}}{2}$ and that B is an s.m.p..
- $A = F_1$, $B = E_j$ $(j = 2, 3, 7)$.
  We find that $P = A$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P) = \sqrt{2}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2\}$, where v_0 is the leading eigenvector of P, $v_1 = B^* v_0$, $v_2 = A^* v_1$.
- $A = F_1$, $B = E_j$ $(j = 5, 6)$.
  We find that $P = A$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P) = \sqrt{2}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6\}$, where v_0 is the leading eigenvector of P, $v_1 = B^* v_0$, $v_2 = A^* v_1$, $v_3 = B^* v_1$, $v_4 = A^* v_3$, $v_5 = B^* v_3$, $v_6 = A^* v_5$.
- $A = F_1$, $B = E_8$.
  We find that $P = AB$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P) = \sqrt{2}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1\}$, where v_0 is the leading eigenvector of P and $v_1 = B^* v_0$.
- $A = F_3$, $B = E_j$ $(j = 1, 4)$.
  Since $\rho(B) = \|B\|_2 = \frac{1+\sqrt{5}}{2}$ and $\|A\|_2 = \sqrt{2}$, we have that $\rho(\mathscr{F}) = \frac{1+\sqrt{5}}{2}$ and that B is an s.m.p..
- $A = F_3$, $B = E_2$.
  We find that $P = (AB)^2 A^2 B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/7} = \left(4\left(2 + \sqrt{3}\right)\right)^{1/7}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\}$, where v_0 is the leading eigenvector of P, $v_1 = A^* v_0$, $v_2 = B^* v_0$, $v_3 = A^* v_1$, $v_4 = A^* v_2$, $v_5 = A^* v_4$, $v_6 = A^* v_5$, $v_7 = B^* v_5$, $v_8 = A^* v_7$, $v_9 = A^* v_8$, $v_{10} = B^* v_9$.
- $A = F_3$, $B = E_3$.
  We find that $P = A^2 B A^3 B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/7} = \left(4\left(2 + \sqrt{2}\right)\right)^{1/7}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$, where v_0 is the leading eigenvector of P, $v_1 = A^* v_0$, $v_2 = B^* v_0$, $v_3 =$

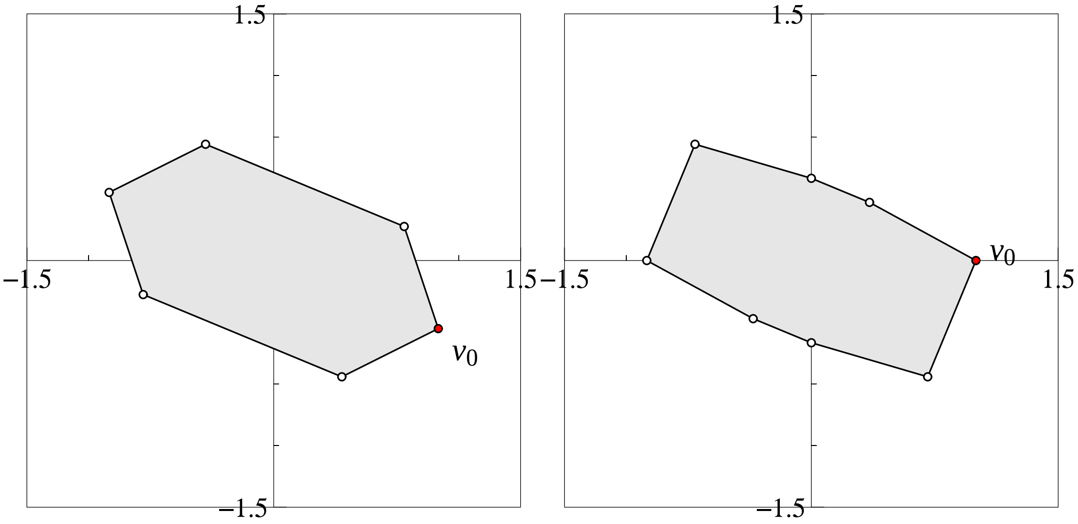

Fig. 8. Polytope norm for the pair $\{A = F_1, B = E_3\}$ (left) and for the pair $A = \{F_1, B = E_5\}$ (right).

$A^* v_2$, $v_4 = A^* v_3$, $v_5 = A^* v_4$, $v_6 = B^* v_5$, $v_7 = A^* v_6$.

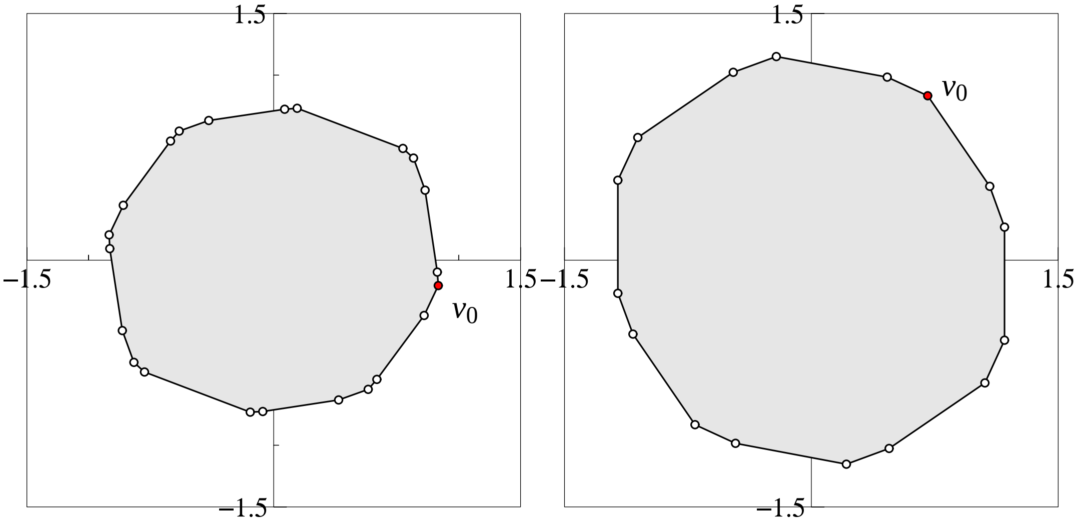

Fig. 9. Polytope norm for the pair $\{A = F_3, B = E_2\}$ (left) and for the pair $\{A = F_3, B = E_3\}$ (right).

- $A = F_3$, $B = E_5$.

  We find that $P = A^3 B^2$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/5} = \left(2\left(2+\sqrt{2}\right)\right)^{1/5}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$, where v_0 is the leading eigenvector of P, $v_1 = B^* v_0$, $v_2 = A^* v_1$, $v_3 = B^* v_1$, $v_4 = A^* v_2$, $v_5 = A^* v_3$, $v_6 = A^* v_4$, $v_7 = A^* v_5$.
- $A = F_3$, $B = E_6$.

  We find that $P = AB^2$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/3} = \left(2+\sqrt{2}\right)^{1/3}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4,$

$v_5, v_6, v_7\}$, where v_0 is the leading eigenvector of P, $v_1 = A^* v_0$, $v_2 = B^* v_0$, $v_3 = A^* v_1$, $v_4 = A^* v_2$, $v_5 = B^* v_2$, $v_6 = A^* v_4$, $v_7 = A^* v_6$.

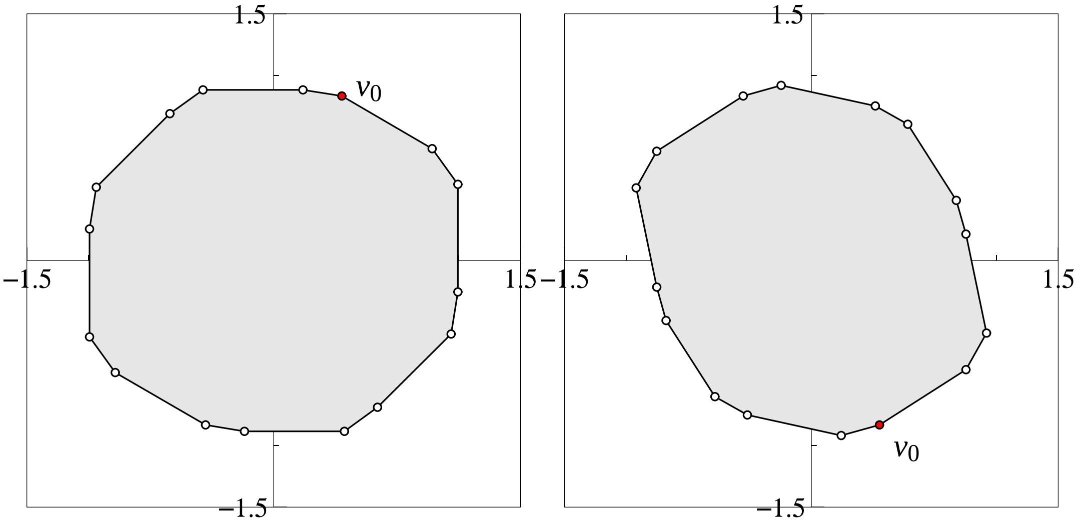

Fig. 10. Polytope norm for the pair $\{A = F_3, B = E_5\}$ (left) and for the pair $\{A = F_3, B = E_6\}$ (right).

- $A = F_3$, $B = E_j$ $(j = 7, 8)$.
  We find that $P = A^2 B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/3} = \left(1 + \sqrt{5}\right)^{1/3}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4\}$, where v_0 is the leading eigenvector of P, $v_1 = A^* v_0$, $v_2 = B^* v_0$, $v_3 = A^* v_2$.
- $A = F_5$, $B = E_j$ $(j = 1, 4)$.
  We find that $P = B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P) = \frac{1+\sqrt{5}}{2}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1\}$, where v_0 is the leading eigenvector of P and $v_1 = A^* v_0$.
- $A = F_5$, $B = E_j$ $(j = 2, 8)$.
  We find that $P = AB$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/2} = \sqrt{3}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1\}$, where v_0 is the leading eigenvector of P and $v_1 = B^* v_0$.
- $A = F_5$, $B = E_j$ $(j = 3, 7)$.
  Since $\rho(B) = \|A\|_*^- = \|B\|_*^- = 1$, we have that $\rho(\mathscr{F}) = 1$ and that B is an s.m.p..
- $A = F_5$, $B = E_j$ $(j = 5, 6)$.
  We find that $P = AB^4$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/5} = 4^{1/5}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5\}$, where v_0 is the leading eigenvector of P, $v_1 = B^* v_0$, $v_2 = B^* v_1$, $v_3 = B^* v_2$, $v_4 = B^* v_3$, $v_5 = B^* v_4$.
- $A = F_8$, $B \in \mathbf{E}$.
  Since $\rho(A) = \|A\|_1 = \|B\|_1 = 2$, we have that $\rho(\mathscr{F}) = 2$ and that A is an s.m.p..

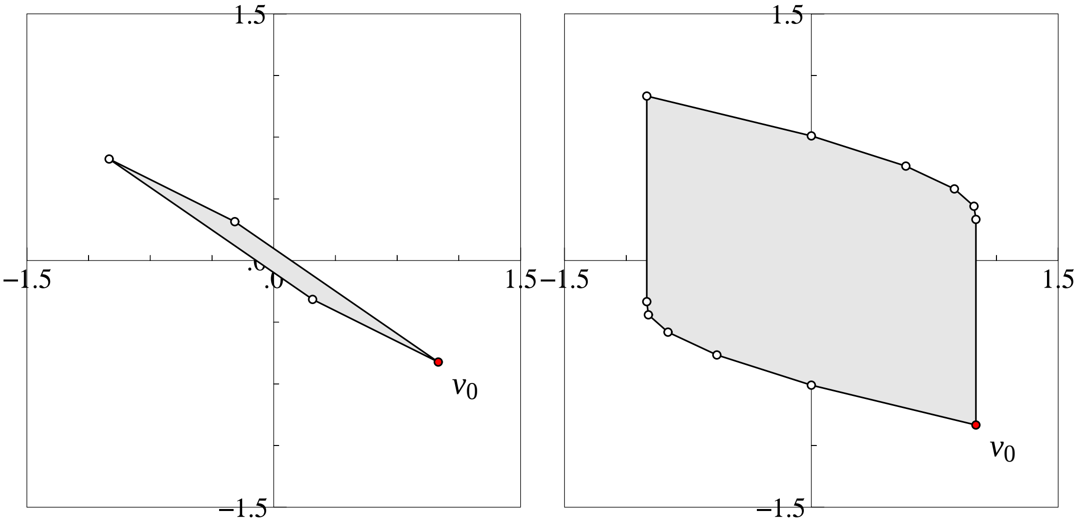

Fig. 11. Polytope norm for the pair $\{A=F_5, B=E_4\}$ (left) and for the pair $\{A=F_5, B=E_5\}$ (right).

The subcase $(n_0, n_1) = (4,4)$.

It corresponds to families of the type $\mathscr{F} = \{F_i, F_j\}$.

- $A=F_8$, $B \in \mathbf{F}$.
  Since $\rho(A) = \|A\|_1 = \|B\|_1 = 2$, we have that $\rho(\mathscr{F}) = 2$ and that A is an s.m.p..

Now it is useful to observe that $P_3 F_1 P_3^{-1} = -F_1$, $P_3 F_3 P_3^{-1} = F_3$, $P_3 F_5 P_3^{-1} = -F_6$ and $P_3 F_8 P_3^{-1} = F_7$. Consequently, when $A=F_i$ $(i=1,3)$, we can restrict the choice of the matrix B within the set $\mathbf{F}'' = \{F_2, F_3, F_4, F_5\}$.

- $A=F_1$, $B=F_j$ $(j=2,3,4)$.
  Since $\rho(A) = \rho(B) = \|A\|_2 = \|B\|_2 = \sqrt{2}$, we have that $\rho(\mathscr{F}) = \sqrt{2}$ and that both A and B are s.m.p.s.
- $A=F_1$, $B=F_5$.
  We find that $P=AB$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/2} = \sqrt{2}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1\}$, where v_0 is the leading eigenvector of P and $v_1 = B^* v_0$.
- $A=F_3$, $B=F_j$ $(j=2,4)$.
  Since $\rho(A) = \rho(B) = \|A\|_2 = \|B\|_2 = \sqrt{2}$, we have that $\rho(\mathscr{F}) = \sqrt{2}$ and that both A and B are s.m.p.s.
- $A=F_3$, $B=F_5$.
  We find that $P=A^2B$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/3} = 4^{1/3}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2, v_3\}$, where v_0 is the leading eigenvector of P, $v_1 = B^* v_0$, $v_2 = A^* v_0, v_3 = A^* v_1$.
- $A=F_5$, $B=F_2$.
  We find that $P=AB^2$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/3} = 4^{1/3}$ and an extremal

polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1, v_2, v_3\}$, where v_0 is the leading eigenvector of P, $v_1 = B^* v_0$, $v_2 = B^* v_1, v_3 = B^* v_2$.

- $A = F_5$, $B = F_4$.
  We find that $P = AB$ is an s.m.p., $\rho(\mathscr{F}) = \rho(P)^{1/2} = \sqrt{2}$ and an extremal polytope norm is given by $\mathscr{P} = \mathrm{co}\,(V, -V)$ with $V = \{v_0, v_1\}$, where v_0 is the leading eigenvector of P and $v_1 = B^* v_0$.
- $A = F_5$, $B = F_6$.
  Since $\|A\|_1 = \|B\|_1 = 2$ and $\rho(AB) = 4$, we have that $\rho(\mathscr{F}) = 2$ and that $P = AB$ is an s.m.p..

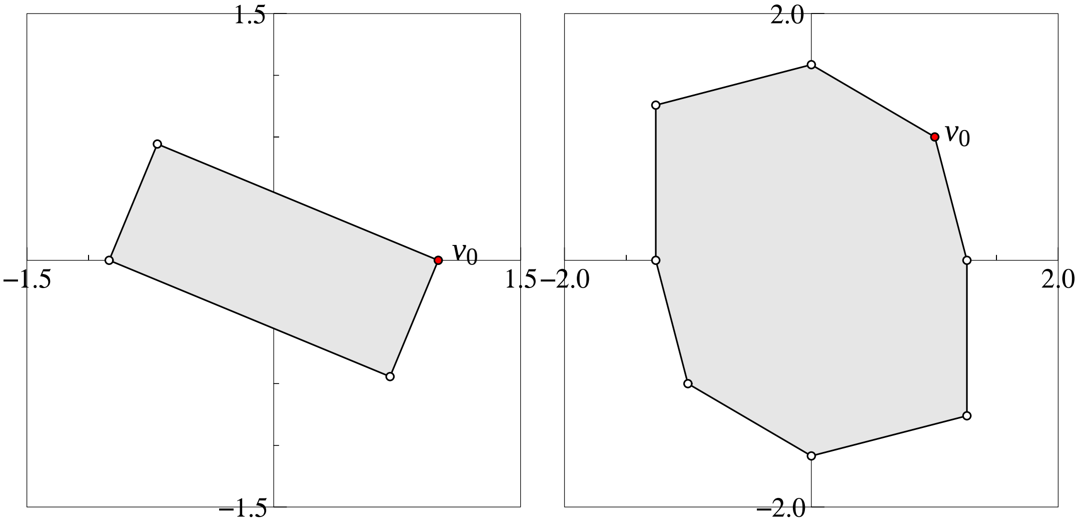

Fig. 12. Polytope norm for the pair $\{A = F_1, B = F_5\}$ (left) and for the pair $\{A = F_3, B = F_5\}$ (right).

4 Conclusions and future work

We have proved the finiteness property for any pair of 2×2-sign matrices. In most non-trivial cases, this has been made possible by detecting an extremal real polytope norm for the family constituted by the two sign-matrices. The finite convergence of the procedure for constructing the unit ball of such a norm, carried out on a case-by-case basis, implies the finiteness property.

Unfortunately, it seems clear that such an approach can hardly been extended to the general case of a pair of sign-matrices of arbitrary dimension. The use of an induction argument on the dimension seems difficult but still needs to be explored.

An algorithm for the construction of an extremal polytope norm is also provided and made publically available.

5 Acknowledgments

A Matlab version of Algorithm 2.1 is available on the webpage of Nicola Guglielmi,

http://univaq.it/-guglielm .

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