

FINDING EXTREMAL COMPLEX POLYTOPE NORMS FOR FAMILIES OF REAL MATRICES*

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Abstract. In this paper we consider finite families \mathcal{F} of real $n \times n$ -matrices. In particular, we focus on the computation of the *joint spectral radius* $\rho(\mathcal{F})$ via the detection of an *extremal norm* in the class of *complex polytope norms*, whose unit balls are *balanced complex polytopes* with a finite *essential system of vertices*. Such a finiteness property is very useful in view of the construction of efficient computational algorithms. More precisely, we improve the results obtained in our previous paper [GWZ05], where we gave some conditions on the family \mathcal{F} which are sufficient to guarantee the existence of an extremal complex polytope norm. Unfortunately, they are such to exclude unnecessarily many interesting cases of real families. Therefore, here we relax somehow the conditions given in [GWZ05], in order to provide a more satisfactory treatment of the real case.

Keywords: Families of matrices, joint spectral radius, extremal norms, complex polytope norms, complex conjugate leading eigenpairs.

1. Introduction. In the last decade, a significant progress has been done within the theory of the *joint spectral radius* of matrix families. As is well known, given a bounded set of real or complex matrices \mathcal{F} , its joint spectral radius $\rho(\mathcal{F})$ determines the maximal growth of all products that can be formed by taking factors in \mathcal{F} . Therefore, its knowledge is important in all applications where the change of status can be described by more than one matrix. For instance, it characterizes the regularity of certain wavelets (see, e.g., [DL92, Mae95, Mae98]), the capacity of codes (see, e.g., [MAS01]), the stability of hybrid systems (see, e.g., [Bar88, Wir98]) or of the numerical solution of ordinary differential equations (see, e.g., [GZ01b]). Further references can be found in [DL01, Wir02].

In the light of this, it clearly appears that efficient methods for the computation of the joint spectral radius of a given set of matrices would be often very welcome. Unfortunately, we are still far from the availability of such efficient methods on a general setting basis. Indeed, the theoretical forecasts in this sense are not well-disposed at all (see, e.g., [TB97]). Nevertheless, an algorithm for efficiently computing lower and upper bounds to $\rho(\mathcal{F})$ has been proposed in [Gri96] and, lately, further promising approaches for the approximation of $\rho(\mathcal{F})$ have been considered (see, e.g., [BN05b, BNT05, BN05a, Pro96, Pro05]).

In [GWZ05], also the authors of the present paper have recently given a contribution in the direction of the computation of $\rho(\mathcal{F})$ by considering special classes of complex matrix families. In particular, they have determined some conditions on the family which are sufficient to guarantee the existence of an extremal *complex polytope norm*, that is a norm whose unit ball is a *balanced complex polytope* with a finite *essential system of vertices* (see [GZ07, VZss]). Such a finiteness property is very useful in view of the construction of algorithms aimed at the actual computation of $\rho(\mathcal{F})$ via the detection of an extremal norm, as is done, for example, in [GZ08].

However, the hypotheses made in [GWZ05] on the family of matrices are such to exclude unnecessarily many interesting cases of real families. In this paper, we confine ourselves to real matrix families and succeed in relaxing somehow the sufficient conditions assumed in [GWZ05], so as to provide a more satisfactory treatment of the real case. As in [GWZ05], under such relaxed conditions, we are able to construct the unit ball of an extremal norm, which is a balanced complex polytope with a finite essential system of vertices.

The organization of the paper is the following. In Section 2 we resume the definitions and the results given in [GWZ05]. The main results of this paper are stated and proved in Section 3. Then, in Section 4 we present a new computational algorithm and an illustrative example. A final discussion is presented in Section 5.

2. The small CPE theorem. In this section we resume the main definitions and results given in [GWZ05].

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As far as the definitions of *joint spectral radius* $\rho(\mathcal{F})$ of a matrix family \mathcal{F} , *nondefective family of matrices*, *extremal norm* and *complex polytope norm*, along with all the necessary related theoretical background, we refer the reader to Sections 2,3 and 4 in [GWZ05].

Here we explicitly recall the so called *Finiteness Conjecture*, arisen from work in [DL92] and stated in [LW95], which involves the following definitions.

DEFINITION 2.1 (Finiteness Property). *A finite family of complex $n \times n$ -matrices $\mathcal{F} = \{A^{(i)}\}_{i \in \mathcal{I}}$ is said to have the finiteness property if there exist $k^* \geq 1$ and a product $\tilde{P} \in \Sigma_{k^*}(\mathcal{F})$ such that*

$$\rho(\mathcal{F}) = \rho(\tilde{P})^{1/k^*}, \quad (2.1)$$

where $\rho(\tilde{P})$ is the spectral radius of the matrix \tilde{P} and, for each $k \geq 1$, $\Sigma_k(\mathcal{F})$ denotes the set of all possible products of length k whose factors are elements of \mathcal{F} , that is $\Sigma_k(\mathcal{F}) = \{A^{(i_1)} \cdots A^{(i_k)} \mid i_1, \dots, i_k \in \mathcal{I}\}$.

DEFINITION 2.2. *If \mathcal{F} is a bounded family of complex $n \times n$ -matrices, any matrix $\tilde{P} \in \Sigma_{k^*}(\mathcal{F})$ satisfying (2.1) for some $k^* \geq 1$ will be called a spectrum-maximizing product (in short, an s.m.p.) for \mathcal{F} .*

Observe that the finiteness property means the existence of at least one s.m.p. \tilde{P} for finite families. In [LW95] some sufficient conditions in terms of extremal norms, guaranteeing that the finiteness property holds, have been given. On the contrary, the Finiteness Conjecture, stating that all finite sets of matrices have the finiteness property, was instead recently disproved in [BM02] and, later, also in [BTV03].

The following definition from [GZ01a] will be useful too.

DEFINITION 2.3. *Assume that \mathcal{F} is a normalized bounded family of complex $n \times n$ -matrices (i.e., $\rho(\mathcal{F}) = 1$) and that there exists a sequence of products $Q_k \in \Sigma_{d_k}(\mathcal{F})$, d_k nondecreasing integers, such that*

$$\lim_{k \rightarrow \infty} Q_k = \tilde{Q}, \quad (2.2)$$

where $\tilde{Q} \in \overline{\Sigma(\mathcal{F})}$ and $\rho(\tilde{Q}) = 1$. Then \tilde{Q} will be called a limit spectrum-maximizing product (l.s.m.p.)

Complex polytope norms play a particular role.

PROPOSITION 2.4. *Let \mathcal{N}_{pol} denotes the set of all possible induced $n \times n$ -matrix complex polytope norms. The spectral radius of a bounded family \mathcal{F} of complex $n \times n$ -matrices is characterized by the equality*

$$\rho(\mathcal{F}) = \inf_{\|\cdot\| \in \mathcal{N}_{pol}} \|\mathcal{F}\|. \quad (2.3)$$

The natural question arises whether a nondefective family admits an extremal complex polytope norm. An important necessary condition for this follows.

THEOREM 2.5. *Let $\mathcal{F} = \{A^{(i)}\}_{1 \leq i \leq m}$ be a finite nondefective family of complex $n \times n$ -matrices and assume that there exists an extremal complex polytope norm $\|\cdot\|_{\mathcal{P}}$. Then \mathcal{F} has at least an s.m.p. \tilde{P} .*

It would be nice to be able to reverse Theorem 2.5, but this has not yet been done unless assuming some additional conditions on the family \mathcal{F} . Given a bounded family $\mathcal{F} = \{A^{(i)}\}_{i \in \mathcal{I}}$ of complex $n \times n$ -matrices with $\rho(\mathcal{F}) > 0$, let us consider the *normalized family* $\hat{\mathcal{F}} = \{\rho(\mathcal{F})^{-1}A^{(i)}\}_{i \in \mathcal{I}}$, whose spectral radius is $\rho(\hat{\mathcal{F}}) = 1$ and, for any vector $x \in \mathbb{C}^n$, consider the set

$$\mathcal{T}[\hat{\mathcal{F}}, x] = \{x\} \cup \{\hat{P}x \mid \hat{P} \in \Sigma(\hat{\mathcal{F}})\},$$

i.e. the *trajectory* obtained by applying all the normalized products \hat{P} of matrices of $\hat{\mathcal{F}}$ to the vector x .

PROPOSITION 2.6. *Let \mathcal{F} be a nondefective bounded family of complex $n \times n$ -matrices and, given a vector $x \in \mathbb{C}^n$, let*

$$\text{span}\left(\mathcal{T}[\hat{\mathcal{F}}, x]\right) = \mathbb{C}^n. \quad (2.4)$$

Then the set

$$\mathcal{S}[\hat{\mathcal{F}}, x] = \overline{\text{span}\left(\mathcal{T}[\hat{\mathcal{F}}, x]\right)}$$

is the unit ball of an extremal norm for \mathcal{F} .

By virtue of the foregoing result, it is interesting to find conditions under which $\mathcal{S}[\hat{\mathcal{F}}, x]$ is generated by a finite number of points of the trajectory $\mathcal{T}[\hat{\mathcal{F}}, x]$. So, if (2.4) holds, the set $\mathcal{S}[\hat{\mathcal{F}}, x]$ is a b.c.p. and, thus, we have an extremal complex polytope norm.

Our attention is focused on families that satisfy some particular properties.

DEFINITION 2.7. *An eigenvector $x \neq 0$ of a matrix P related to an eigenvalue λ with $|\lambda| = \rho(P)$ is said to be a leading eigenvector of P .*

DEFINITION 2.8. *Let \mathcal{F} be a nondefective bounded family of complex $n \times n$ -matrices. A leading eigenvector $x \neq 0$ of either an s.m.p. \tilde{P} of \mathcal{F} or of an l.s.m.p. \tilde{Q} of the normalized family $\hat{\mathcal{F}}$ is said to be leading eigenvector of \mathcal{F} (and of $\hat{\mathcal{F}}$ too).*

THEOREM 2.9. *Let \mathcal{F} be a (possibly infinite) nondefective bounded family of complex $n \times n$ -matrices. Then there exists an l.s.m.p. \tilde{Q} for the normalized family $\hat{\mathcal{F}}$ and, consequently, \mathcal{F} has at least one leading eigenvector.*

DEFINITION 2.10. *Let \mathcal{F} be a family of complex $n \times n$ -matrices. A set $\mathcal{X} \subset \mathbb{C}^n$ is said to be \mathcal{F} -cyclic if for any pair $(x, y) \in \mathcal{X} \times \mathcal{X}$ there exist $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \cdot |\beta| = 1$ and two (finite) normalized products $\hat{P}, \hat{Q} \in \Sigma(\hat{\mathcal{F}})$ such that $y = \alpha \hat{P}x$ and $x = \beta \hat{Q}y$.*

REMARK 2.1. *Since $|\alpha| \cdot |\beta| = 1$, the normalized products $\hat{P}\hat{Q}$ and $\hat{Q}\hat{P}$ determined in the above definition are s.m.p. of the normalized family $\hat{\mathcal{F}}$ and the set \mathcal{X} is necessarily included in the set \mathcal{L} of the leading eigenvectors of the family \mathcal{F} .*

DEFINITION 2.11. *A nondefective bounded family \mathcal{F} of complex $n \times n$ -matrices is said to be asymptotically simple if the set \mathcal{L} of its leading eigenvectors is finite (modulo scalar nonzero factors) and \mathcal{F} -cyclic.*

We shall say that a matrix Q is an infinite power of another matrix P if it is a limit point of the sequence $\{P^k\}_{k \geq 1}$. Observe that any eigenvalue λ of an infinite power Q of a matrix P satisfies either $|\lambda| = 1$ or $\lambda = 0$, since these are the only two possible limit values of the numeric power sequence $\{|\mu|^k\}_{k \geq 1}$ whenever $|\mu| \leq 1$. Moreover, given a nondefective matrix P with $\rho(P) = 1$, there exists at least an infinite power Q of P with an eigenvalue $\lambda = 1$, whose multiplicity is equal to the sum of the multiplicities of all the eigenvalues μ of P with $|\mu| = 1$. This easily follows from the fact that the power sequence $\{\mu^k\}_{k \geq 1}$ has the limit point 1 whenever $|\mu| = 1$ (see, for example, [HW79]).

REMARK 2.2. *It follows from the above observations that, for a (nondefective) asymptotically simple family \mathcal{F} , each s.m.p. \tilde{P} and each l.s.m.p. \tilde{Q} have only one leading eigenvector (modulo scalar nonzero factors). Otherwise there would exist at least one l.s.m.p. of the normalized family $\hat{\mathcal{F}}$, obtained as an infinite power, with an eigenspace of dimension ≥ 2 related to the eigenvalue $\lambda = 1$. This would contradict the finiteness (modulo scalar nonzero factors) of the set \mathcal{L} of leading eigenvectors.*

Observe that all the cyclic permutations of a product P have the same eigenvalues with the same multiplicities. Thus, if $\tilde{P} = A^{(i_{k^*})} \dots A^{(i_1)}$ is an s.m.p. for a family \mathcal{F} , then each of its cyclic permutations

$$A^{(i_s)} \dots A^{(i_1)} A^{(i_{k^*})} \dots A^{(i_{s+1})}, \quad s = 1, \dots, k^* - 1,$$

still is an s.m.p. for \mathcal{F} , along with all the powers of \tilde{P} and their cyclic permutations.

DEFINITION 2.12. *Let \mathcal{F} be a family of complex $n \times n$ -matrices. An s.m.p. \tilde{P} is said to be minimal if it is not a power of another s.m.p. of \mathcal{F} .*

It is clear that, for any s.m.p. \tilde{P} of a family \mathcal{F} , it holds that either \tilde{P} is minimal or \tilde{P} is a power of another s.m.p., which is minimal.

We have the following characterization of asymptotically simple families.

PROPOSITION 2.13. *A nondefective bounded family \mathcal{F} of complex $n \times n$ -matrices is asymptotically simple if and only if it has a minimal s.m.p. \tilde{P} with only one leading eigenvector (modulo scalar nonzero factors) such that the set \mathcal{L} of the leading eigenvectors of \mathcal{F} is equal to the set of the leading eigenvectors of \tilde{P} and of its cyclic permutations.*

The following definition selects a particular class of asymptotically simple families.

DEFINITION 2.14. *A nondefective bounded family \mathcal{F} of complex $n \times n$ -matrices is said to be absolutely asymptotically simple if it is asymptotically simple and has a unique minimal s.m.p. \tilde{P} (modulo cyclic permutations).*

It is clear that, for absolutely asymptotically simple families, the unique minimal s.m.p. \tilde{P} coincides with the minimal s.m.p. given by the characterizing Proposition 2.13. Moreover, the cardinality of the set \mathcal{L} of its leading eigenvectors (modulo scalar nonzero factors) is equal to the number of factors of \tilde{P} .

The following theorem is the main result of [GWZ05].

THEOREM 2.15 (small CPE theorem). *Assume that a finite family $\mathcal{F} = \{A^{(i)}\}_{1 \leq i \leq m}$ of complex $n \times n$ -matrices is nondefective and asymptotically simple. Moreover, let $x \neq 0$ be a leading eigenvector of \mathcal{F} and assume that (2.4) is satisfied. Then the set*

$$\partial \mathcal{S}[\hat{\mathcal{F}}, x] \cap \mathcal{T}[\hat{\mathcal{F}}, x] \quad (2.5)$$

is finite modulo scalar factors of unitary modulus. As a consequence, there exist a finite number of normalized products $\hat{P}^{(1)}, \dots, \hat{P}^{(s)} \in \Sigma(\hat{\mathcal{F}})$ such that

$$\mathcal{S}[\hat{\mathcal{F}}, x] = \text{absco}\left(\{x, \hat{P}^{(1)}x, \dots, \hat{P}^{(s)}x\}\right), \quad (2.6)$$

so that $\mathcal{S}[\hat{\mathcal{F}}, x]$ is a b.c.p.

Remark that, if all the matrices of the family \mathcal{F} are real and if also the starting leading eigenvector x is real, then Theorem 2.15 determines a classical polytope in \mathbb{R}^n .

The next results are useful for a deeper understanding of the structure of the b.c.p. $\mathcal{S}[\hat{\mathcal{F}}, x]$ obtained under the hypotheses of Theorem 2.15.

THEOREM 2.16. *Let the hypotheses of Theorem 2.15 hold. Then each leading eigenvector ξ of \mathcal{F} in the set $\Xi = \mathcal{L} \cap \partial \mathcal{S}[\hat{\mathcal{F}}, x]$ satisfies one of the following two statements:*

- (a) ξ is a vertex of the b.c.p. $\mathcal{S}[\hat{\mathcal{F}}, x]$;
- (b) there exist $s \geq 2$ vertices ξ_1, \dots, ξ_s of the b.c.p. $\mathcal{S}[\hat{\mathcal{F}}, x]$ such that

$$\xi_1, \dots, \xi_s \in \Xi \quad \text{and} \quad \xi \in \text{absco}\left(\{\xi_1, \dots, \xi_s\}\right). \quad (2.7)$$

COROLLARY 2.17. *Let the hypotheses of Theorem 2.15 hold and, moreover, let the family \mathcal{F} be absolutely asymptotically simple. Then all the leading eigenvectors of \mathcal{F} (in the set $\Xi = \mathcal{L} \cap \partial \mathcal{S}[\hat{\mathcal{F}}, x]$) are vertices of the b.c.p. $\mathcal{S}[\hat{\mathcal{F}}, x]$.*

3. Improving the small CPE theorem for real families. Unfortunately, the hypotheses assumed by Theorem 2.15 are such to exclude some cases of real families that, on the contrary, clearly admit an extremal complex polytope norm which could be determined by a suitable modification of the procedure based on the construction of the trajectory $\mathcal{T}[\hat{\mathcal{F}}, x]$. We illustrate this fact by means of the following example.

EXAMPLE 3.1. Consider the real 2×2 -matrix family $\mathcal{F} = \{A, B\}$, where

$$A = \begin{bmatrix} \cos(1) & -\sin(1) \\ \sin(1) & \cos(1) \end{bmatrix} \quad \text{and} \quad B = \beta \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & 0 \end{bmatrix} \quad \text{with } 0 < \beta \leq 1.$$

The eigenvalues of A are e^{-i} and e^i with corresponding eigenvectors $x = [1 \ i]^T$ and $\bar{x} = [1 \ -i]^T$, respectively. The eigenvalues of B are $\frac{\beta\sqrt{2}}{2}$ and 0 . Thus $\rho(A) = 1$ and $\rho(B) = \frac{\beta\sqrt{2}}{2}$. Now consider the b.c.p.

$$\mathcal{P} = \text{absco}(\{x, \bar{x}\}), \quad (3.1)$$

whose boundary $\partial \mathcal{P}$ intersects \mathbb{R}^2 on the unit circle

$$\mathcal{C} = \left\{ [x_1 \ x_2]^T \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1 \right\}. \quad (3.2)$$

It is immediately seen that

$$A \mathcal{P} = \mathcal{P} \quad \text{and} \quad B \mathcal{P} = \text{absco}([\beta \ 0]^T) \subset \mathcal{P},$$

where we mean by $A \mathcal{P}$ the set $\{y = Ax : x \in \mathcal{P}\}$ and similarly by $B \mathcal{P}$. Consequently

$$\|A\|_{\mathcal{P}} = 1, \quad \|B\|_{\mathcal{P}} = \beta, \quad \rho(\mathcal{F}) = \|\mathcal{F}\|_{\mathcal{P}} = 1.$$

Consequently, whenever $\beta < 1$, the matrix A is the unique minimal s.m.p. of \mathcal{F} and all the l.s.m.p.'s of \mathcal{F} are (infinite) powers of A .

On the contrary, for $\beta = 1$ it is not difficult to prove that, whereas A remains the unique minimal s.m.p., there exist some l.s.m.p.'s other than the (infinite) powers of A such as the matrices $A^\infty B$ and BA^∞ , where

$$A^\infty = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix},$$

which is an infinite power of A . In fact, we have

$$A^\infty B \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}^T = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}^T \quad \text{and} \quad BA^\infty [1 \ 0]^T = [1 \ 0]^T.$$

Remark that these two l.s.m.p.'s are cyclic permutations of each other and that they have only one real leading eigenvector. In any case, for all $\beta \leq 1$, according to Remark 2.2, the identity matrix I is an l.s.m.p., so that all the vectors of \mathbb{C}^2 are leading eigenvectors of \mathcal{F} . Moreover, it is clear that the set of leading eigenvectors of \mathcal{F} is not \mathcal{F} -cyclic. Therefore, the hypotheses of Theorem 2.15 are definitely violated.

Indeed, the set $\mathcal{S}[\mathcal{F}, x]$ is not a b.c.p. In fact, it holds that

$$Bx = \begin{bmatrix} \frac{\beta\sqrt{2}(1+i)}{2} \\ 0 \end{bmatrix}^T \quad \text{and} \quad A^k Bx = \frac{\beta\sqrt{2}(1+i)}{2} [\cos(k) \ \sin(k)]^T, \quad k \geq 1.$$

All these vectors lie on the circle

$$\mathcal{C}_\beta = \frac{\beta\sqrt{2}(1+i)}{2} \mathcal{C},$$

where \mathcal{C} is defined by (3.2), and form a set which is dense in \mathcal{C}_β . On the other hand,

$$BA^k Bx = \begin{bmatrix} \frac{\beta(\cos(k) + \sin(k))(1+i)}{2} \\ 0 \end{bmatrix}^T \in \text{absco}(\mathcal{C}_\beta),$$

and, hence, we can conclude that

$$\mathcal{S}[\mathcal{F}, x] = \text{absco}\{\{x\} \cup \mathcal{C}_\beta\},$$

that is not a b.c.p. Note that infinitely many vectors of the trajectory $\mathcal{T}[\mathcal{F}, x]$, namely $A^k Bx$ for $k \geq 0$, which are not proportional to one another, lie on $\partial \mathcal{S}[\mathcal{F}, x]$. Analogous conclusions hold for the conjugate set $\mathcal{S}[\mathcal{F}, \bar{x}]$. On the other hand, it is interesting to observe that

$$\mathcal{P} = \overline{\text{absco}(\mathcal{T}[\mathcal{F}, x] \cup \mathcal{T}[\mathcal{F}, \bar{x}])}, \quad (3.3)$$

where \mathcal{P} is the b.c.p. given by (3.1), and that, in the light of the previous analysis, we have

$$\partial \mathcal{P} \cap (\mathcal{T}[\mathcal{F}, x] \cup \mathcal{T}[\mathcal{F}, \bar{x}]) = \{x, \bar{x}\}$$

(modulo scalar factors of unitary modulus), whenever $\beta < 1$. On the contrary, for $\beta = 1$ the infinitely many vectors $A^k Bx$ (and $A^k B\bar{x}$), $k \geq 0$, which are not proportional to one another, lie on $\partial \mathcal{P}$. \diamond

The foregoing example suggests naturally how to try to relax the hypotheses of Theorem 2.15 in order to accommodate the case of a real family such that the minimal s.m.p.'s have a pair of complex conjugate leading eigenvectors instead of a single real leading eigenvector. In order to do this, we need to modify some of the definitions given in [GWZ05] and reported in Section 2, and to give some new ones.

DEFINITION 3.1. *Let \mathcal{F} be a nondefective bounded family of complex $n \times n$ -matrices and let $x \neq 0$ be a leading eigenvector of \mathcal{F} . Then x is said to be a standard leading eigenvector of \mathcal{F} (and of $\hat{\mathcal{F}}$ too) if it is a leading eigenvector of a minimal s.m.p. \tilde{P} of \mathcal{F} , whereas it is said to be a limit leading eigenvector of \mathcal{F} if it is a leading eigenvector of an l.s.m.p. (possibly, an s.m.p.) \tilde{Q} of the normalized family $\hat{\mathcal{F}}$, but is not a leading eigenvector of any minimal s.m.p..*

Now, we modify Definition 2.11 for real families.

DEFINITION 3.2. *Let \mathcal{F} be a nondefective bounded family of real $n \times n$ -matrices and \mathcal{E} be the set of its standard leading eigenvectors. The family \mathcal{F} is said to be asymptotically simple if \mathcal{E} is finite (modulo scalar nonzero factors) and the following properties hold:*

- *If \mathcal{E} is a real set, then:*
 - (i-r) *\mathcal{E} is \mathcal{F} -cyclic.*
 - (ii-r) *If \tilde{Q} is an l.s.m.p. of $\hat{\mathcal{F}}$, then the set of its leading eigenvectors is included in \mathcal{E} .*
- *If \mathcal{E} is not a real set, then:*
 - (i) *\mathcal{E} is self-conjugate, that is $\mathcal{E} = \mathcal{E}_1 \cup \overline{\mathcal{E}}_1$, and \mathcal{E}_1 is \mathcal{F} -cyclic.*
 - (ii) *If \tilde{Q} is an l.s.m.p. of $\hat{\mathcal{F}}$, then the set of its leading eigenvectors lies in a subspace of dimension 2 and includes a complex conjugate pair of elements of \mathcal{E} .*

Observe that, when the leading eigenvectors are all real, there is no change with respect to Definition 2.11.

REMARK 3.1. *If a family \mathcal{F} of real $n \times n$ -matrices is asymptotically simple and \tilde{P} is a minimal s.m.p., then \tilde{P} has a unique leading eigenvector if the set \mathcal{E} is real, whereas \tilde{P} has a unique pair of complex conjugate leading eigenvectors if the set \mathcal{E} is not real.*

REMARK 3.2. *For a nondefective asymptotically simple normalized family $\hat{\mathcal{F}}$ such that the set \mathcal{E} is not real, there exists an l.s.m.p. \tilde{Q} , obtained as an infinite power (possibly, a finite power) of a minimal s.m.p. \tilde{P} , with an eigenspace of dimension 2 related to the eigenvalue $\lambda = 1$.*

The following characterization of asymptotically simple real families holds.

PROPOSITION 3.3. *A nondefective bounded family \mathcal{F} of real $n \times n$ -matrices is asymptotically simple if and only if one of the following situations occurs:*

- 1) *\mathcal{F} fulfils property (ii-r) of Definition 3.2 and has a minimal s.m.p. \tilde{P} with a unique real leading eigenvector such that the set \mathcal{E} of the standard leading eigenvectors of \mathcal{F} is equal to the set of the leading eigenvectors of \tilde{P} and of its cyclic permutations.*
- 2) *\mathcal{F} fulfils property (ii) of Definition 3.2 and has a minimal s.m.p. \tilde{P} with a unique pair of complex conjugate leading eigenvectors such that the set \mathcal{E} of the standard leading eigenvectors of \mathcal{F} is equal to the set of the leading eigenvectors of \tilde{P} and of its cyclic permutations.*

Proof.

1) The proof of this occurrence is given in [GWZ05].

2) *Necessity.* Let $x_1, \dots, x_s, \bar{x}_1, \dots, \bar{x}_s \in \mathbb{C}^n$ form a set of distinct representatives (modulo scalar nonzero factors) of all the standard leading eigenvectors of \mathcal{F} . Since \mathcal{F} is asymptotically simple, they are finitely many and, for any $i = 1, \dots, s$, there exist $\alpha_i, \beta_i \in \mathbb{C}$ with $|\alpha_i| \cdot |\beta_i| = 1$ and two (finite) normalized products $\hat{P}_i, \hat{Q}_i \in \Sigma(\hat{\mathcal{F}})$ such that

$$x_{i+1} = \alpha_i \hat{P}_i x_i \quad \text{and} \quad x_i = \beta_i \hat{Q}_i x_{i+1}$$

and (by reality of \mathcal{F}),

$$\bar{x}_{i+1} = \bar{\alpha}_i \hat{P}_i \bar{x}_i \quad \text{and} \quad \bar{x}_i = \bar{\beta}_i \hat{Q}_i \bar{x}_{i+1}$$

where, conventionally, $x_{s+1} = x_1$. Therefore, we obtain that $x_1 = \alpha \hat{P}x_1 = \beta \hat{Q}x_1$ and $\bar{x}_1 = \bar{\alpha} \hat{P}\bar{x}_1 = \bar{\beta} \hat{Q}\bar{x}_1$, where $\alpha = \alpha_1 \dots \alpha_s$, $\beta = \beta_1 \dots \beta_s$ and $\hat{P} = \hat{P}_s \dots \hat{P}_1$, $\hat{Q} = \hat{Q}_1 \dots \hat{Q}_s$, with $|\alpha| \cdot |\beta| = 1$. Now, since $\rho(\hat{P}) \leq 1$ and $\rho(\hat{Q}) \leq 1$, it follows that $|\alpha| = |\beta| = 1$, which implies $\rho(\hat{P}) = 1$ and $\rho(\hat{Q}) = 1$.

So we can conclude that the matrix $\tilde{P} = \tilde{P}_s \dots \tilde{P}_1$ is an s.m.p. of \mathcal{F} such that the set of the leading eigenvectors of \tilde{P} and of its cyclic permutations includes (and thus is equal to) the set \mathcal{E} of the standard leading eigenvectors of \mathcal{F} . Since it is not restrictive assuming \tilde{P} to be minimal, the proof is concluded.

Sufficiency. Assume that there is a (minimal) s.m.p. \tilde{P} with only 2 complex conjugate leading eigenvectors and that the set \mathcal{V} of the leading eigenvectors of \tilde{P} and of its cyclic permutations (modulo scalar nonzero factors) coincides with the set \mathcal{E} of standard leading eigenvectors of \mathcal{F} . Therefore, since \mathcal{V} may be generated (through multiplication by nonzero complex numbers) by $2p$ elements, where p denotes the number of factors of \tilde{P} , and since $\mathcal{V} = \mathcal{V}_1 \cup \overline{\mathcal{V}_1}$, where the sets \mathcal{V}_1 and $\overline{\mathcal{V}_1}$ are clearly \mathcal{F} -cyclic, the proof is complete. \square

LEMMA 3.4. *Let the nondefective bounded family \mathcal{F} of real $n \times n$ -matrices be asymptotically simple and let the set \mathcal{E} of its standard leading eigenvectors be non-real. Then the set \mathcal{L} of all the leading eigenvectors of \mathcal{F} is given by*

$$\mathcal{L} = \bigcup_{i=1}^p \text{span}(\{x_i, \bar{x}_i\}),$$

where $\{x_1, \dots, x_p\}$ is a set of linearly independent representatives of \mathcal{E}_1 , introduced in Definition 3.2 - (i).

Proof. As observed in Remark 3.2, for a nondefective asymptotically simple normalized family $\hat{\mathcal{F}}$, there exists an l.s.m.p. \hat{Q} , which is an infinite power (possibly, a finite power) of a minimal s.m.p. \hat{P} , with an eigenspace of dimension 2 related to the eigenvalue $\lambda = 1$. Such eigenspace is clearly spanned by the pair of complex conjugate leading eigenvectors of the minimal s.m.p. \hat{P} . Therefore, by virtue of Proposition 3.3, the proof is complete. \square

The foregoing result implies that the set of the distinct representatives (modulo scalar nonzero factors) of all the leading eigenvectors of \mathcal{F} is not finite.

The definition of *absolutely asymptotically simple* family remains formally unchanged (see Definition 2.14) and, again, for such kind of families, the unique minimal s.m.p. \tilde{P} coincides with the minimal s.m.p. given by the characterizing Proposition 3.3.

Moreover, the cardinality of the set \mathcal{E} of the standard leading eigenvectors (modulo scalar nonzero factors) is either the number of factors of \tilde{P} (in the case that \mathcal{E} is real) or twice such a number (in the case that \mathcal{E} is not real).

In the sequel, for a real family \mathcal{F} and for a non-zero complex conjugate pair of vectors $x, \bar{x} \in \mathbb{C}^n$ (with $\Im(x) \neq 0$), we will consider the *trajectory* obtained by applying all the normalized products $\hat{P} \in \hat{\mathcal{F}}$ to x and \bar{x} , that is

$$\mathcal{T}[\hat{\mathcal{F}}, x, \bar{x}] = \mathcal{T}[\hat{\mathcal{F}}, x] \cup \mathcal{T}[\hat{\mathcal{F}}, \bar{x}] = \{x, \bar{x}\} \cup \left\{ \{\hat{P}x, \hat{P}\bar{x}\} \mid \hat{P} \in \Sigma(\hat{\mathcal{F}}) \right\}, \quad (3.4)$$

and the set

$$\mathcal{S}[\hat{\mathcal{F}}, x, \bar{x}] = \overline{\text{absco}\left(\mathcal{T}[\hat{\mathcal{F}}, x, \bar{x}]\right)}. \quad (3.5)$$

Similarly to Proposition 2.6, we have that, if \mathcal{F} is nondefective and if

$$\text{span}\left(\mathcal{T}[\hat{\mathcal{F}}, x, \bar{x}]\right) = \mathbb{C}^n, \quad (3.6)$$

the set $\mathcal{S}[\hat{\mathcal{F}}, x, \bar{x}]$ is the unit ball of an extremal norm. Therefore, our aim is to find conditions under which $\mathcal{S}[\hat{\mathcal{F}}, x, \bar{x}]$ is generated by a finite number of points of the trajectory $\mathcal{T}[\hat{\mathcal{F}}, x, \bar{x}]$, in order that the set $\mathcal{S}[\hat{\mathcal{F}}, x, \bar{x}]$ be a b.c.p..

In any case, since \mathcal{F} is real, the set $\mathcal{S}[\hat{\mathcal{F}}, x, \bar{x}]$ is self-conjugate. Some specific properties of self-conjugate b.c.p.'s may be found in [VZss].

The proof of the following lemma is analogous to that of Lemma 5.19 in [GWZ05].

LEMMA 3.5. *Let $\mathcal{F} = \{A^{(i)}\}_{1 \leq i \leq m}$ be a nondefective finite family of real $n \times n$ -matrices and, given a non-real non-zero vector $x \in \mathbb{C}^n$, assume that (3.6) holds and that the (bounded) set $\partial \mathcal{S}[\hat{\mathcal{F}}, x, \bar{x}] \cap \mathcal{T}[\hat{\mathcal{F}}, x, \bar{x}]$ is not finite (modulo scalar factors of unitary modulus). Then there exist two conjugate sequences of distinct vectors $\{x^{(k)}\}$ and $\{\bar{x}^{(k)}\}$ with $x^{(1)} = x$ and $\bar{x}^{(1)} = \bar{x}$ such that, for all $k \geq 1$,*

$$x^{(k)}, \bar{x}^{(k)} \in \partial \mathcal{S}[\hat{\mathcal{F}}, x, \bar{x}] \cap \mathcal{T}[\hat{\mathcal{F}}, x, \bar{x}]$$

and

$$x^{(k+1)} = \hat{A}^{(\ell_k)} x^{(k)} \quad \text{and} \quad \bar{x}^{(k+1)} = \hat{A}^{(\ell_k)} \bar{x}^{(k)} \quad \text{for some } \ell_k \in \{1, \dots, m\},$$

where $\hat{A}^{(i)} = A^{(i)} / \rho(\mathcal{F}) \in \hat{\mathcal{F}}$, $1 \leq i \leq m$, and such that, whenever $k \neq h$,

$$x^{(k)} \neq ux^{(h)} \quad \text{and} \quad \bar{x}^{(k)} \neq u\bar{x}^{(h)} \quad \text{for all } u \in \mathbb{C} \text{ with } |u| = 1.$$

Eventually, we are in a position to state the main result of this paper, which extends the validity of the Small CPE Theorem proved in [GWZ05]. However, so far we were not able to prove it unless under the following technical hypothesis, even if we strongly believe that it holds even without assuming it (see also the forthcoming Example 3.2).

HYPOTHESIS 3.1. *If the set \mathcal{E} of the standard leading eigenvectors of the family \mathcal{F} is non-real, then the pair of leading eigenvalues $(e^{i\theta}, e^{-i\theta})$ of any minimal s.m.p. of the normalized family $\hat{\mathcal{F}}$ is such that the numbers θ and π are rationally independent.*

The next proposition illustrates the practical meaning of Hypothesis 3.1.

PROPOSITION 3.6. *Let the finite family \mathcal{F} of real $n \times n$ -matrices be asymptotically simple and let the set \mathcal{E} of its standard leading eigenvectors be non-real. Then Hypothesis 3.1 is equivalent to requiring that every s.m.p. of \mathcal{F} (even if non-minimal) only has a pair of complex conjugate standard leading eigenvectors.*

Proof. Given a minimal s.m.p. \hat{P} of $\hat{\mathcal{F}}$, we have that θ and π are rationally dependent if and only if there exists a positive integer k such that the power \hat{P}^k , which is an s.m.p. of $\hat{\mathcal{F}}$, has the eigenvalue $\lambda = 1$ of multiplicity 2 with infinitely many corresponding eigenvectors (i.e., all the vectors belonging to $\text{span}(\{x, \bar{x}\})$, x and \bar{x} being the leading eigenvectors of \hat{P}). \square

The following lemma plays an important role in the proof of the subsequent main result (Theorem 3.8).

LEMMA 3.7. *Assume that a nondefective finite family $\mathcal{F} = \{A^{(i)}\}_{1 \leq i \leq m}$ of real $n \times n$ -matrices is asymptotically simple, let Hypothesis 3.1 hold and let $\{x, \bar{x}\}$ be a pair of complex conjugate standard leading eigenvectors of \mathcal{F} . Moreover, assume that, for some matrix $\hat{S} \in \Sigma(\hat{\mathcal{F}})$, the vectors $y = \hat{S}x$ and $\bar{y} = \hat{S}\bar{x}$ are leading eigenvectors of \mathcal{F} .*

Then y and \bar{y} are leading eigenvectors of standard type.

Proof. We observe that $y, \bar{y} \in \partial \mathcal{S}[\hat{\mathcal{F}}, x, \bar{x}]$ and that, by Lemma 3.4, $y, \bar{y} \in \text{span}(\{z, \bar{z}\})$ for some standard leading eigenvector z , that is

$$y = \alpha z + \beta \bar{z}, \quad \bar{y} = \bar{\beta} z + \bar{\alpha} \bar{z}, \tag{3.7}$$

for suitable $\alpha, \beta \in \mathbb{C}$. Moreover, we can assume that $z, \bar{z} \in \partial \mathcal{S}[\hat{\mathcal{F}}, x, \bar{x}]$ and, by virtue of \mathcal{F} -cyclicity, that there exists $\hat{Q} \in \Sigma_k(\hat{\mathcal{F}})$ (for some integer k) such that $x = \hat{Q}z$, so that

$$y = \hat{S}\hat{Q}z, \quad \bar{y} = \hat{S}\hat{Q}\bar{z}. \tag{3.8}$$

Then consider a linear transformation (represented by the non-singular matrix $T \in \mathbb{C}^{n,n}$) which transforms z into $e^{(1)}$ and \bar{z} into $e^{(2)}$, $e^{(1)}$ and $e^{(2)}$ being the first two vectors of the canonical basis of \mathbb{C}^n . After setting

$$C^{(i)} = T A^{(i)} T^{-1},$$

we introduce the family of complex matrices $\mathcal{G} = \{C^{(i)}\}_{1 \leq i \leq m}$, which is such that $\rho(\mathcal{G}) = \rho(\mathcal{F})$, and denote by $\hat{\mathcal{G}}$ the associated normalized family. By formulae (3.7), we have that

$$Ty = [\alpha \ \beta \ 0 \ \dots \ 0]^T \quad \text{and} \quad T\bar{y} = [\bar{\beta} \ \bar{\alpha} \ 0 \ \dots \ 0]^T$$

and, therefore, by setting $\hat{M} = T\hat{S}\hat{Q}T^{-1}$, formulae (3.8) imply

$$\hat{M} = \begin{bmatrix} \alpha & \bar{\beta} & \cdot & \dots & \cdot \\ \beta & \bar{\alpha} & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

If $|\alpha| + |\beta| < 1$, then y, \bar{y} are strictly internal to $\text{absco}(\{z, \bar{z}\}) \subseteq \mathcal{S}[\hat{\mathcal{F}}, x, \bar{x}]$, which gives a contradiction. Thus

$$|\alpha| + |\beta| \geq 1. \quad (3.9)$$

According to Proposition 3.3, z and \bar{z} are the leading eigenvectors of a minimal s.m.p. of $\hat{\mathcal{F}}$, say $\hat{P}_\circ \in \Sigma_p(\hat{\mathcal{F}})$, with leading eigenvalues $e^{\pm i\theta_0}$. Now consider the corresponding product $\hat{N} \in \Sigma_p(\hat{\mathcal{G}})$ defined as $\hat{N} = T\hat{P}_\circ T^{-1}$,

$$\hat{N} = \begin{bmatrix} e^{i\theta_0} & 0 & 0 & \dots & 0 \\ 0 & e^{-i\theta_0} & 0 & \dots & 0 \\ 0 & 0 & \cdot & \dots & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdot & \dots & \cdot \end{bmatrix},$$

and the power sequence \hat{N}^ℓ . By Hypothesis 3.1, we can conclude (see, e.g., [HW79]) that there exists a suitable subsequence \hat{N}^{ℓ_m} which converges to the matrix

$$\hat{N}_\infty = \begin{bmatrix} e^{-i \arg(\alpha)} & 0 & 0 & \dots & 0 \\ 0 & e^{i \arg(\alpha)} & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

Then we consider the matrix

$$\hat{O} = \hat{N}_\infty \hat{M} = \begin{bmatrix} |\alpha| & e^{-i \arg(\alpha)} \bar{\beta} & \cdot & \dots & \cdot \\ e^{i \arg(\alpha)} \beta & |\alpha| & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix},$$

which belongs to $\overline{\Sigma(\hat{\mathcal{G}})}$ and whose eigenvalues are $|\alpha| \pm |\beta|$. Since $\hat{\mathcal{G}}$ is normalized, it follows that

$$|\alpha| + |\beta| \leq 1. \quad (3.10)$$

In conclusion, combining (3.9) and (3.10), we get $|\alpha| + |\beta| = 1$. Therefore, \hat{O} is an l.s.m.p. for \mathcal{G} and, hence, the corresponding matrix $O = T^{-1}\hat{O}T$ is an l.s.m.p. for \mathcal{F} . Finally, property (ii) in Definition 3.2 yields

$$|\alpha| = 1 \quad \text{and} \quad \beta = 0.$$

Therefore, by (3.7), y and \bar{y} are standard leading eigenvectors. \square

THEOREM 3.8 (extended small CPE theorem). *Assume that a nondefective finite family $\mathcal{F} = \{A^{(i)}\}_{1 \leq i \leq m}$ of real $n \times n$ -matrices is asymptotically simple. If the set \mathcal{E} of its standard leading eigenvectors is real, then Theorem 2.15 applies. Otherwise, let Hypothesis 3.1 hold and let $\{x, \bar{x}\}$ be a pair of complex conjugate standard leading eigenvectors of \mathcal{F} such that (3.6) holds. Then the set*

$$\partial \mathcal{S}[\hat{\mathcal{F}}, x, \bar{x}] \cap \mathcal{T}[\hat{\mathcal{F}}, x, \bar{x}] \quad (3.11)$$

is finite modulo scalar factors of unitary modulus. As a consequence, there exist a finite number of normalized products $\hat{P}^{(1)}, \dots, \hat{P}^{(s)} \in \Sigma(\hat{\mathcal{F}})$ such that

$$\mathcal{S}[\hat{\mathcal{F}}, x, \bar{x}] = \text{absco} \left(\{x, \bar{x}, \hat{P}^{(1)}x, \hat{P}^{(1)}\bar{x}, \dots, \hat{P}^{(s)}x, \hat{P}^{(s)}\bar{x}\} \right), \quad (3.12)$$

so that $\mathcal{S}[\hat{\mathcal{F}}, x, \bar{x}]$ is a b.c.p.

Proof. Assume that the set \mathcal{E} of the standard leading eigenvectors is not real and let $\{x, \bar{x}\}$ be a complex conjugate pair of them. According to Proposition 3.3, x and \bar{x} are the leading eigenvectors of a minimal s.m.p. \hat{P} of $\hat{\mathcal{F}}$. Now consider

$$\Xi = \mathcal{E} \cap \partial \mathcal{S}[\hat{\mathcal{F}}, x, \bar{x}]. \quad (3.13)$$

Since the family \mathcal{F} is asymptotically simple and $\mathcal{S}[\hat{\mathcal{F}}, x, \bar{x}]$ is the unit ball of a norm, the set Ξ is finite modulo scalar factors of unitary modulus and not empty. Now assume, by contradiction, that the set

$$\partial \mathcal{S}[\hat{\mathcal{F}}, x, \bar{x}] \cap \mathcal{T}[\hat{\mathcal{F}}, x, \bar{x}],$$

even if considered modulo scalar factors of unitary modulus, is not finite, so that Lemma 3.5 can be applied to obtain the sequences $x^{(k)}, \bar{x}^{(k)} \in \partial \mathcal{S}[\hat{\mathcal{F}}, x, \bar{x}] \cap \mathcal{T}[\hat{\mathcal{F}}, x, \bar{x}]$ with $x^{(1)} = x$ and $\bar{x}^{(1)} = \bar{x}$. Therefore, there exists $j \geq 1$ such that

$$x^{(j+1)} \notin \Xi \quad \text{and} \quad x^{(i)} \in \Xi \cap \mathcal{T}[\hat{\mathcal{F}}, x] \quad \text{for all } i \leq j, \quad (3.14)$$

and, similarly,

$$\bar{x}^{(j+1)} \notin \Xi \quad \text{and} \quad \bar{x}^{(i)} \in \Xi \cap \mathcal{T}[\hat{\mathcal{F}}, \bar{x}] \quad \text{for all } i \leq j.$$

Since $\Sigma(\hat{\mathcal{F}})$ is bounded, the resulting sequence of normalized matrix products $\hat{B}^{(k)} = \hat{A}^{(\ell_{k-1})} \dots \hat{A}^{(\ell_{j+1})}$ such that $x^{(k)} = \hat{B}^{(k)}x^{(j+1)}$ has a subsequence $\{\hat{B}^{(k_s)}\}_{s \geq 1}$ that converges to a limit point \hat{B} in $\Sigma(\hat{\mathcal{F}})$. Therefore, also the subsequences of vectors $\{x^{(k_s)}\}_{s \geq 1}$ and $\{\bar{x}^{(k_s)}\}_{s \geq 1}$ have limit points

$$v = \hat{B}x^{(j+1)} = \hat{B}\hat{A}^{(\ell_j)} \dots \hat{A}^{(\ell_1)}x \quad \text{and} \quad \bar{v} = \hat{B}\bar{x}^{(j+1)} = \hat{B}\hat{A}^{(\ell_j)} \dots \hat{A}^{(\ell_1)}\bar{x}. \quad (3.15)$$

For each $s \geq 1$ there exists a matrix $\hat{R}^{(s)} \in \Sigma(\hat{\mathcal{F}})$ such that

$$\hat{B}^{(k_{s+1})} = \hat{R}^{(s)}\hat{B}^{(k_s)}.$$

Again by the boundedness of $\Sigma(\hat{\mathcal{F}})$, the sequence $\{\hat{R}^{(s)}\}_{s \geq 1}$ has a limit point \hat{R} in $\Sigma(\hat{\mathcal{F}})$. By passing to the limit, we can conclude that $\hat{B} = \hat{R}\hat{B}$ and, thus, $v = \hat{R}v$ and $\bar{v} = \hat{R}\bar{v}$.

In other words, \hat{R} is an l.s.m.p. of $\hat{\mathcal{F}}$ and v and \bar{v} are leading eigenvectors of \mathcal{F} . Thus, with $y = v$ and $\hat{S} = \hat{B}\hat{A}^{(\ell_j)} \dots \hat{A}^{(\ell_1)}$, Lemma 3.7 implies that v and \bar{v} are standard leading eigenvectors. Therefore, by the assumed \mathcal{F} -ciclicity and since $v, \bar{v} \in \partial \mathcal{S}[\hat{\mathcal{F}}, x, \bar{x}]$, there exists $\hat{Q} \in \Sigma_k$ (for some integer k) and some $\varphi \in (0, 2\pi]$ such that

$$x = e^{i\varphi}\hat{Q}v \quad \text{or} \quad x = e^{i\varphi}\hat{Q}\bar{v}.$$

Consequently, by (3.15), we obtain

$$x^{(j+1)} = e^{i\varphi} \hat{A}^{(\ell_j)} \dots \hat{A}^{(\ell_1)} \hat{Q} \hat{B} x^{(j+1)} \quad \text{or} \quad x^{(j+1)} = \left(\hat{A}^{(\ell_j)} \dots \hat{A}^{(\ell_1)} \hat{Q} \hat{B} \right)^2 x^{(j+1)}.$$

Therefore, since $\hat{A}^{(\ell_j)} \dots \hat{A}^{(\ell_1)} \hat{Q} \hat{B} \in \overline{\Sigma(\hat{\mathcal{F}})}$, in both cases the vector $x^{(j+1)}$ is a leading eigenvector of \mathcal{F} . Thus, with $y = x^{(j+1)}$ and $\hat{\mathcal{S}} = \hat{A}^{(\ell_j)} \dots \hat{A}^{(\ell_1)}$, Lemma 3.7 can be applied again to conclude that $x^{(j+1)}$ is a standard leading eigenvector, which contradicts (3.14). \square

Observe that the family $\mathcal{F} = \{A, B\}$ of Example 3.1 fits perfectly the hypotheses of Theorem 3.8 for all $\beta < 1$, but not for $\beta = 1$. In fact, for $\beta = 1$ the set (3.11) is not finite modulo scalar factors of unitary modulus, even if (3.12) holds all the same.

The next example illustrates the case of a family that satisfies the hypotheses of Lemma 3.7 and Theorem 3.8 but Hypothesis 3.1. It shows that, whereas the thesis of Lemma 3.7 fails to hold, that of Theorem 3.8 is satisfied. Therefore, we suspect that the thesis of Theorem 3.8 could be obtained in another way, without passing necessarily through Lemma 3.7.

EXAMPLE 3.2. Consider the real 2×2 -matrix family $\mathcal{F} = \{A, B\}$, where

$$A = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \frac{11}{20} & \frac{11}{20} \\ -\frac{11}{20} & -\frac{11}{20} \end{bmatrix}.$$

The eigenvalues of A are $e^{-i\frac{2\pi}{3}}$ and $e^{i\frac{2\pi}{3}}$ with corresponding eigenvectors $x = [1 \ i]^T$ and $\bar{x} = [1 \ -i]^T$, respectively, so that Hypothesis 3.1 clearly does not hold. However, it is easy to see that \mathcal{F} is absolutely asymptotically simple with $\rho(\mathcal{F}) = 1$, A being the unique minimal s.m.p., that $\partial\mathcal{S}[\mathcal{F}, x] \cap \mathcal{S}[\mathcal{F}, x]$ is finite (modulo scalar factors of unitary modulus) and that

$$\mathcal{S}[\mathcal{F}, x, \bar{x}] = \text{absco}\left(\{x, \bar{x}, Bx, ABx, A^2Bx\}\right).$$

For this purpose, note that $A^3 = I$ (identity matrix) and $B^2 = O$ (zero matrix) and that the range of B is a one-dimensional subspace proportional to the real vector $[\frac{11}{20} \ -\frac{11}{20}]^T$.

Thus, since all nonzero vectors of \mathbb{C}^2 are leading eigenvectors of \mathcal{F} and since Bx , ABx and A^2Bx are not of standard type, we can conclude that Lemma 3.7 is not true and that, on the contrary, the thesis of Theorem 3.8 does hold all the same. \diamond

The next results, analogous to Theorem 2.16 and to Corollary 2.17, respectively, provide a deeper understanding of the structure of the b.c.p. $\mathcal{S}[\hat{\mathcal{F}}, x, \bar{x}]$ obtained under the hypotheses of Theorem 3.8.

THEOREM 3.9. *Let the hypotheses of Theorem 3.8 hold and let the set \mathcal{E} of the standard leading eigenvectors of \mathcal{F} be non-real. Then each eigenvector $\xi \in \Xi = \mathcal{E} \cap \partial\mathcal{S}[\hat{\mathcal{F}}, x, \bar{x}]$ satisfies one of the following two statements:*

- (a) ξ is a vertex of the b.c.p. $\mathcal{S}[\hat{\mathcal{F}}, x, \bar{x}]$.
- (b) There exist $s \geq 2$ vertices ξ_1, \dots, ξ_s of the b.c.p. $\mathcal{S}[\hat{\mathcal{F}}, x, \bar{x}]$ such that

$$\xi_1, \dots, \xi_s \in \Xi \quad \text{and} \quad \xi \in \text{absco}\left(\{\xi_1, \dots, \xi_s\}\right). \quad (3.16)$$

Proof. Consider a standard leading eigenvector $\xi \in \Xi$ and assume that it is not a vertex of the b.c.p. $\mathcal{S}[\hat{\mathcal{F}}, x, \bar{x}]$. Then there must exist $s \geq 2$ vertices ξ_1, \dots, ξ_s of $\mathcal{S}[\hat{\mathcal{F}}, x, \bar{x}]$ such that

$$\xi = \sum_{i=1}^s \lambda_i \xi_i \quad \text{with} \quad \lambda_i \neq 0, \quad i = 1, \dots, s, \quad \text{and} \quad \sum_{i=1}^s |\lambda_i| = 1. \quad (3.17)$$

Since ξ is a standard leading eigenvector of \mathcal{F} , there exists an s.m.p. \tilde{P} of $\hat{\mathcal{F}}$ such that $\tilde{P}\xi = u\xi$ with $|u| = 1$. Thus, denoting by $\|\cdot\|$ the complex polytope norm determined by $\mathcal{S}[\hat{\mathcal{F}}, x, \bar{x}]$, for any $k \geq 1$ we have

$$1 = \|\xi\| = \|\tilde{P}^k \xi\| \leq \sum_{i=1}^s |\lambda_i| \cdot \|\tilde{P}^k \xi_i\|.$$

Since $\|\tilde{P}^k \xi_i\| \leq 1$, in view of (3.17) we can claim that $\|\tilde{P}^k \xi_i\| = 1$, that is

$$\tilde{P}^k \xi_i \in \partial \mathcal{S}[\hat{\mathcal{F}}, x, \bar{x}], \quad i = 1, \dots, s.$$

By Theorem 3.8, for any fixed i , the set of vectors $\{\tilde{P}^k \xi_i\}_{k=0}^{\infty}$ is finite (modulo scalar factors of unitary modulus). Hence there exist integers ℓ, m such that

$$\tilde{P}^{\ell+m} \xi_i = \gamma \tilde{P}^{\ell} \xi_i, \quad \text{with } |\gamma| = 1.$$

This implies that $\tilde{P}^{\ell} \xi_i$ is a standard leading eigenvector of \tilde{P}^m .

Moreover, since all the vertices of the b.c.p. $\mathcal{S}[\hat{\mathcal{F}}, x, \bar{x}]$ obviously belong to $\mathcal{T}[\hat{\mathcal{F}}, x, \bar{x}]$ (modulo scalar factors of unitary modulus), there exist finite normalized products $\hat{S}_i \in \Sigma(\hat{\mathcal{F}})$ such that

$$\xi_i = \hat{S}_i x.$$

This assumption is not restrictive because the other possibility, that is $\xi_i = \hat{S}_i \bar{x}$, would lead to the same conclusions.

Furthermore, \mathcal{F} -cyclicity implies that there exist finite normalized products $\hat{R}_i, \hat{U}_i \in \Sigma(\hat{\mathcal{F}})$ and complex numbers r_i and u_i with $|r_i| = |u_i| = 1$ such that

$$\begin{cases} \xi_i &= r_i \hat{S}_i \hat{R}_i \xi \\ \text{or} & \\ \xi_i &= r_i \hat{S}_i \hat{R}_i \bar{\xi} \end{cases} \quad (3.18)$$

$$\begin{cases} \xi &= u_i \hat{U}_i \tilde{P}^{\ell} \xi_i \\ \text{or} & \\ \xi &= u_i \hat{U}_i \tilde{P}^{\ell} \bar{\xi}_i \end{cases} \quad (3.19)$$

Using the first of (3.18) and the first of (3.19) we obtain that ξ_i is a standard leading eigenvector of \mathcal{F} whereas, using the second of (3.18) and the second of (3.19), we obtain that $\bar{\xi}_i$ (and thus ξ_i) is a standard leading eigenvector of \mathcal{F} . Similarly, using the first of (3.18) and the second of (3.19) (or, specularly, the second of (3.18) and the first of (3.19)) we obtain $\xi_i = \gamma_i \hat{S}_i \hat{R}_i \hat{U}_i \tilde{P}^{\ell} \bar{\xi}_i$ for some γ_i with $|\gamma_i| = 1$ and then, in turn, $\xi_i = (\hat{S}_i \hat{R}_i \hat{U}_i \tilde{P}^{\ell})^2 \xi_i$, which means again that ξ_i is a standard leading eigenvector of \mathcal{F} . Therefore, we conclude that $\xi_i \in \Xi$. \square

COROLLARY 3.10. *Let the hypotheses of Theorem 3.8 hold and let the set \mathcal{E} of the standard leading eigenvectors of \mathcal{F} be non-real. Moreover, let the family \mathcal{F} be absolutely asymptotically simple. Then each eigenvector $\xi \in \Xi = \mathcal{E} \cap \partial \mathcal{S}[\hat{\mathcal{F}}, x, \bar{x}]$ is a vertex of the b.c.p. $\mathcal{S}[\hat{\mathcal{F}}, x, \bar{x}]$.*

Proof. Assume, by contradiction, that there exists a standard leading eigenvector $\xi \in \Xi$ which is not a vertex of the b.c.p. $\mathcal{S}[\hat{\mathcal{F}}, x, \bar{x}]$. Then it necessarily satisfies statement (b) of Theorem 3.9.

On the other hand, there exists a unique normalized minimal s.m.p. \tilde{P} such that $\tilde{P}\xi = u\xi$ with $|u| = 1$ (see Proposition 3.3). Therefore, for each ξ_i appearing in statement (b) there exists a proper normalized right factor \tilde{P}_i of the s.m.p. \tilde{P} such that $\xi_i = u_i \tilde{P}_i \xi$ or $\xi_i = u_i \tilde{P}_i \bar{\xi}$ with $|u_i| = 1$.

Thus we obtain

$$\xi_i \in \text{absco}\left(\{\tilde{P}_i \xi_1, \dots, \tilde{P}_i \xi_s\}\right) \quad \text{or} \quad \xi_i \in \text{absco}\left(\{\tilde{P}_i \bar{\xi}_1, \dots, \tilde{P}_i \bar{\xi}_s\}\right).$$

Now, since the essential system of vertices of a b.c.p. is unique modulo scalar factors of unitary modulus and since ξ_i is a vertex of $\mathcal{S}[\hat{\mathcal{F}}, x, \bar{x}]$, it necessarily holds that, for all $j = 1, \dots, s$,

$$\xi_i = v_j \tilde{P}_i \xi_j \quad \text{or} \quad \xi_i = v_j \tilde{P}_i \bar{\xi}_j \quad \text{with } |v_j| = 1.$$

In particular, for $j = i$, we obtain

$$\xi_i = v_i \tilde{P}_i \xi_i \quad \text{with } |v_i| = 1 \quad \text{or} \quad \xi_i = \tilde{P}_i^2 \xi_i,$$

which, in both cases, implies that the proper normalized right factor \tilde{P}_i of \tilde{P} is itself an s.m.p., against the uniqueness of \tilde{P} (modulo cyclic permutations). \square

4. Applications of the extended small CPE theorem. Although our results are mostly theoretical in nature, they have potential impact on applications. One of these would lie in the fact that, if there is prior knowledge that a certain set \mathcal{F} has an extremal complex polytope norm, then one could devise algorithms for the computation of $\rho(\mathcal{F})$ that rely on the computation of the extremal points of the unit ball of the norm.

Now we propose a suitable modification of the algorithm presented in [GZ08], which was based on the small CPE theorem proved in [GWZ05]. This new version allows us to construct the unit ball of an extremal complex polytope norm for a nondefective finite real family $\mathcal{F} = \{A^{(i)}\}_{1 \leq i \leq m}$ which satisfies the hypotheses of Theorem 3.8 in the case that the set \mathcal{E} of the standard leading eigenvectors is not real.

ALGORITHM 4.1.

(Step 1) Choose a candidate s.m.p. $P_k \in \Sigma_k(\mathcal{F})$ (for some k), which we assume to have a pair $\{x, \bar{x}\}$ of complex conjugate leading eigenvectors.

(Step 2) Compute $\rho = \rho(P_k)^{1/k}$ and define the scaled family

$$\mathcal{F}^* = \rho^{-1} \mathcal{F} \quad (\text{which is such that } \rho(\mathcal{F}^*) \geq 1).$$

(Step 3) Set $\mathcal{W}^{(0)} = \mathcal{V}^{(0)} = \mathcal{X}^{(0)} = \{x, \bar{x}\}$, $\mathcal{P}^{(0)} = \text{absco}(\mathcal{X}^{(0)})$ and $s = 1$.

(Step 4) Compute the set of vectors

$$\mathcal{V}^{(s)} = \mathcal{F}^* \left(\mathcal{W}^{(s-1)} \right).$$

(Step 5) If $\mathcal{V}^{(s)} \subset \mathcal{P}^{(s-1)}$ then STOP

(Step 6) Set $\mathcal{P}^{(s)} = \text{absco}(\mathcal{X}^{(s-1)} \cup \mathcal{V}^{(s)})$ and compute an essential system of vertices $\mathcal{X}^{(s)}$ of $\mathcal{P}^{(s)}$ such that

$$\mathcal{X}^{(s)} \subseteq \mathcal{X}^{(s-1)} \cup \mathcal{V}^{(s)}.$$

(Step 7) Set $\mathcal{W}^{(s)} = \mathcal{X}^{(s)} \cap \mathcal{V}^{(s)}$, $s = s + 1$ and Goto (Step 4).

The procedure produces a (possibly finite) sequence of self-conjugate absolutely convex sets $\mathcal{P}^{(s)}$. If it halts at (Step 5) for some s^* and if $\text{span}(\mathcal{X}^{(s^*-1)}) = \mathbb{C}^n$, then necessarily $\rho(\mathcal{F}^*) = 1$, so that \mathcal{F}^* is nothing but the normalized family $\hat{\mathcal{F}}$. Moreover, the self-conjugate b.c.p. $\mathcal{P}^{(s^*-1)}$ is equal to $\mathcal{S}[\hat{\mathcal{F}}, x, \bar{x}]$, which determines an extremal norm for \mathcal{F} , and we have that

$$\rho(\mathcal{F}) = \rho(P_k)^{1/k}.$$

We conclude the paper by illustrating the foregoing algorithm with an example.

EXAMPLE 4.1. We consider the real 4×4 -matrix family $\mathcal{F} = \{A, B\}$, where

$$A = \begin{bmatrix} -3 & -2 & 1 & 2 \\ -2 & 0 & -2 & 1 \\ 1 & 3 & -1 & -5 \\ -3 & -3 & -2 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & -3 & -1 \\ -4 & -2 & -1 & -4 \\ -1 & 0 & -1 & 2 \\ -1 & -2 & -1 & 2 \end{bmatrix}.$$

Step 1. On the basis of a preliminary computational investigation, we guess that $P_2 = AB$ is a reasonable candidate s.m.p. for the family \mathcal{F} . We compute the two leading complex conjugate eigenpairs of P_2 , that is,

$$\lambda_{1,2} = 10.874286706162435289 \pm 17.646835410096406617i$$

and

$$x, \bar{x} = \begin{bmatrix} 0.72792977781254290782 \mp 0.34062961167742906135i \\ 0.20390080201908339851 \pm 0.41457724740357478799i \\ -0.56613846958271645283 \pm 1.2278515180664638032i \\ 1 \end{bmatrix}.$$

Step 2. We scale the family \mathcal{F} by $\rho(P_2)^{1/2} = |\lambda_{1,2}|^{1/2} = 4.5528302832023213335$, so as to obtain

$$\mathcal{F}^* = \{A^*, B^*\} = \left\{ \frac{A}{\rho(P_2)^{1/2}}, \frac{B}{\rho(P_2)^{1/2}} \right\}.$$

Step 3. The two starting vectors are $v_1 = x$ and $\bar{v}_1 = \bar{x}$.

We set $\mathcal{W}^{(0)} = \mathcal{V}^{(0)} = \mathcal{X}^{(0)} = \{v_1, \bar{v}_1\}$ and $\mathcal{P}^{(0)} = \text{absco}(\mathcal{X}^{(0)})$.

Main loop. By applying $s = 6$ iterations of the loop (Step 4)–(Step 7) we are able to find a complex polytope which is mapped by the scaled family \mathcal{F}^* into itself.

Since $\text{span}(\mathcal{P}^{(6)}) = \mathbb{C}^4$, we can conclude that $\mathcal{P}^{(6)} = \text{absco}(\mathcal{X}^{(6)})$ is a self-conjugate b.c.p., which determines an extremal norm for \mathcal{F} , and that $\rho(\mathcal{F}) = |\lambda_{1,2}|^{1/2} = 4.5528302832023213335$.

The final polytope $\mathcal{P}^{(6)}$ turns out to have the following essential system of vertices

$$\mathcal{X}^{(6)} = \{v_i\}_{i=1}^{19} \cup \{\bar{v}_i\}_{i=1}^{19},$$

where $v_2 = A^*v_1, v_3 = B^*v_1, v_4 = A^*v_2, v_5 = B^*v_2, v_6 = B^*v_3, v_7 = A^*v_4, v_8 = A^*v_5, v_9 = A^*v_6, v_{10} = B^*v_4, v_{11} = B^*v_5, v_{12} = A^*v_7, v_{13} = A^*v_{10}, v_{14} = B^*v_7, v_{15} = B^*v_8, v_{16} = B^*v_9, v_{17} = A^*v_{15}, v_{18} = A^*v_{16}, v_{19} = A^*v_{17}$.

The computations¹ have been performed using both the software *Matlab*, exploiting the package *Sedumi* (see [Stu99]) for the computation of the complex polytope norms at (Step 5), and the software *Mathematica* (see [Wol05]) with a 30-digit accuracy. The computation of the complex polytope norm has been restated as an optimization problem with quadratic constraints, for the solution of which we have used the function *NMinimize*. \diamond

5. Concluding discussion. In this section we discuss the results of this paper in the light of the existing literature.

First of all, we observe that there exist families of real matrices which have an extremal complex polytope norm but do not admit any extremal real polytope norm. For example, any family $\mathcal{F} = \{A_\theta\}$ consisting of a single rotation matrix

$$A_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix},$$

with θ rationally independent on π , does not admit any extremal real polytope norm (see [BW92]). However, it admits a complex polytope norm with unit ball given by $\mathcal{P} = \text{absco}(\{z_\theta, \bar{z}_\theta\})$, z_θ being a leading eigenvector of A_θ . This justifies the use of complex polytope norms and related algorithms also for families of real matrices.

¹The whole example (coded in Matlab language) is available on the homepage of the first author, <http://univaq.it/~guglielm/>.

Concerning our main result, that is Theorem 3.8, we remark that the proof is based on Lemma 3.5, which makes use of Hypothesis 3.1 in an essential way. Since the lemma is not valid if Hypothesis 3.1 does not hold (see Example 3.2), it seems that, in order to extend the validity of Theorem 3.8 to the this case, a different proof should be found. On the other hand, Hypothesis 3.1 appears to be generic in the sense that the measure of the set of excluded cases is zero. Nevertheless, we consider such an extension of the theorem an important theoretical issue.

Our approach for the computation of the joint spectral radius passes through the construction of the unit ball of a polytope extremal norm. In order to illustrate its theoretical relevance, we recall its applications to the zero-stability analysis of variable stepsize BDF formulæ[GZ01b] and to the asymptotic stability analysis of one-step methods for the numerical approximation of delay differential equations [GZ03]. Finally, in the recent paper [GCCZ], we have proved that every pair of 2×2 binary sign-matrices, that is, with entries in $\{-1, 0, 1\}$, has the finiteness property. This supports the conjecture by Blondel and Jüngers [JB08] that this holds in general for all pairs of sign-matrices of any dimension. In turn, this fact would have the consequence that the finiteness property holds true for all families of rational matrices.

Now we discuss the computational relevance of Algorithm 4.1. It is clear that, at its current state, the algorithm is designed to verify that a candidate product $P \in \Sigma(\mathcal{F})$ is an s.m.p. for a non-defective family \mathcal{F} . This is possible under certain assumptions, the main of which is asymptotic simplicity, which cannot be checked a priori. Nevertheless, if P is an s.m.p., the algorithm always converges. Indeed, the mentioned assumptions provide sufficient conditions for convergence in a finite number of steps.

We conclude by remarking that a different, quite interesting, approach in order to approximate the joint spectral radius of a finite family \mathcal{F} of real matrices has been proposed recently by Protasov [Pro96, Pro05]. The main idea is still based on the property that any irreducible family \mathcal{F} has an extremal norm and that its unit ball is a centrally symmetric invariant compact set for \mathcal{F} . The algorithm constructs an *almost invariant* set \mathcal{R} , that is, such that

$$\min_{\lambda > 0} \text{dist}(\text{co}(\mathcal{F}(\mathcal{R})), \lambda \mathcal{R})$$

be sufficiently small, where dist denotes some distance between sets and $\text{co}(\mathcal{F}(\mathcal{R}))$ denotes the convex hull of the union of the sets $A(\mathcal{R})$ for all $A \in \mathcal{F}$. If this holds, the minimizer λ^* provides a good approximation of $\rho(\mathcal{F})$.

Protasov's algorithm is able to reach a given accuracy ε in polynomial time with respect to $1/\varepsilon$. Its implementation is based on the recursive application of \mathcal{F} to a sequence of real polytopes which are defined in the following way. The first polytope \mathcal{R}_0 is chosen, for example, as the unit ball of the 1-norm. Then the polytope \mathcal{R}_{m+1} is given either by $\text{co}(\mathcal{F}(\mathcal{R}_m))$ if the number of its vertices does not exceed a certain bound $v(\varepsilon)$ or, otherwise, by a polytope with at most $v(\varepsilon)$ vertices chosen in such a way that

$$(1 + \varepsilon) \text{co}(\mathcal{F}(\mathcal{R}_m)) \subset \mathcal{R}_{m+1} \subset \text{co}(\mathcal{F}(\mathcal{R}_m)).$$

At the m -th step the approximation to $\rho(\mathcal{F})$ is given by $\sqrt[m]{d_m}$, where d_m is the radius of the smallest circle including \mathcal{R}_m .

Differently from our approach, in general the algorithm proposed by Protasov does not show convergence in a finite number of steps but, on the other hand, it does not require a guess for an s.m.p. (which is crucial for Algorithm 4.1) and also provides estimates of the accuracy obtained at every iteration.

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