NUMERICAL PERIODIC ORBITS OF NEUTRAL DELAY DIFFERENTIAL EQUATIONS

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Abstract. This paper deals with the long-time behaviour of numerical solutions of neutral delay differential equations that have stable hyperbolic periodic orbits. It is shown that Runge–Kutta discretizations of such equations have attractive invariant closed curves which approximate the periodic orbit with the full order of the method, in spite of the lack of a finite-time smoothing property of the flow.

1. Introduction. It is a basic question in the dynamics of numerical methods for evolution equations as to whether invariant sets of the equation have their counterpart in the discretization, and what order of approximation, if any, there is between the invariant sets of the continuous dynamical system and its discretization. These questions have been studied for many kinds of evolution equations, invariant objects, and numerical methods; see, e.g., the reviews and numerous references in [13,16].

In particular, for ordinary differential equations with a hyperbolic periodic orbit, it has been shown in [2,3,5] and [16 Sect. 6.6] that one-step methods have invariant closed curves which approximate the periodic orbit. For delay differential equations such results are shown in [6,11], but no results in this direction have so far been given for neutral delay differential equations.

Here, we analyze Runge-Kutta discretizations of systems of neutral delay differential equations of the form

\[ x'(t) = f(x(t), x(t - \tau)) + A x'(t - \tau), \quad t \geq 0, \]

with general (not necessarily consistent) initial functions. The particular form with a constant matrix A (of spectral radius < 1) is chosen for convenience, not as a necessity.

We assume that the equation has a stable hyperbolic periodic orbit and ask for its approximation by an attractive invariant curve of the numerical method. The analysis of the analogous problem for delay differential equations in [11] used

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the smoothing property of the flow as an important tool, which is not available here. Nevertheless, for the neutral equation (1) we will obtain an approximation result of the same type: an attractive invariant closed curve of the discretization that approximates the periodic orbit with the full order $O(h^p)$ of the Runge-Kutta method.

In Section 2 we put up the framework for this paper. Section 3 gives the main result on the attractive invariant curve of the Runge–Kutta discretization, and Section 4 illustrates the result by numerical experiments.

2. Framework.

2.1. Notation. We rewrite the neutral problem (1) in the following form:

\[
\begin{aligned}
  x'(t) &= u(t) \\
  0 &= u(t) - f(x(t), x(t-\tau)) - A u(t-\tau)
\end{aligned}
\]

with initial data $x(t) = x^0(t)$ and $u(t) = u^0(t)$ for $-\tau \leq t \leq 0$. We do not require the condition $(x^0)'(t) = u^0(t)$.

We assume that $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is a smooth function, and $A$ is a constant matrix with spectral radius $\rho(A) < 1$.

The initial data $x^0, u^0$ are in $C = C([-\tau, 0], \mathbb{R}^d)$, the Banach space of vector-valued continuous functions on $[-\tau, 0]$ equipped with the maximum norm,

\[
\|y\| = \max_{-\tau \leq s \leq 0} |y(s)|,
\]

where $| \cdot |$ is an arbitrary but fixed norm on $\mathbb{R}^d$.

Under the given assumptions, the system (2) has a unique solution (see, e.g., [10]). We define the solution segment $(x^\theta, u^\theta) \in C \times C$ by

\[
x^\theta(t) = x(t+\theta), \quad u^\theta(t) = u(t+\theta) \quad \text{for } \theta \in [-\tau, 0].
\]

We denote by $S^t = (S^t_x, S^t_u)$ the flow map of (2), which associates the state $(x^\theta, u^\theta)$ to the initial data $(x^0, u^0)$. This is a Fréchet differentiable semigroup

\[
S^t : C \times C \to C \times C;
\]

in particular, $S^{t+s} = S^t \circ S^s$ for all $t, s \geq 0$.

2.2. Assumptions. We assume that the system (2) has a stable hyperbolic periodic orbit, that is,

(i) Eq. (2) has a nonconstant periodic solution (with period $\omega$)

\[
(\bar{x}, \bar{u}) : \mathbb{R} \to \mathbb{R}^{2d},
\]

where $\bar{u}(t) = \bar{x}'(t)$;

(ii) the Fréchet derivative of the period map

\[
D_x S^\omega_\omega (\bar{x}^0, \bar{u}^0)
\]

has a spectrum lying in a complex disk $\{ |\lambda| \leq \rho \}$ of radius $\rho < 1$, with the exception of the simple eigenvalue 1.
2.3. Numerical method. We apply a Runge–Kutta method with constant step-size
\[ h = \frac{\tau}{\nu} \]
with a positive integer \( \nu \)
in the “natural” way (see e.g. \[\Pi\]). We assume that
the Runge–Kutta method is of order \( p \),
i.e., the error of the method applied to ordinary differential equations over bounded
time intervals is of size \( O(h^p) \). We use the following notation:
\( a_{ij}, b_i \) and \( c_i \) (\( i, j = 1, \ldots, m \)) are the coefficients of the Runge-Kutta method;
\( y_n, v_n \) denote the Runge-Kutta approximations to \( x(nh), u(nh) \);
\( Y_m, V_m \) denotes the pair of \( m \)th stage values computed at the \( m \)th step;
The Runge–Kutta formula applies as
\[
\begin{aligned}
Y_{ni} &= y_n + h \sum_{j=1}^{m} a_{ij} V_{nj} \\
0 &= v_{ni+1} - f \left( y_{ni+1}, y_{ni+1}^{\nu} \right) - A v_{ni+1}^{\nu} \\
0 &= V_{ni} - f \left( Y_{ni}, Y_{ni}^{\nu} \right) - A V_{ni}^{\nu},
\end{aligned}
\]
(3)
where the starting values, for negative indices \( n \), are assumed to lie on the exact
initial data:
\[
Y_{ni} = x^0(n h + c_i h) \\
V_{ni} = v^0(n h + c_i h) \quad n < 0.
\]
2.4. Continuous formulation of the Runge-Kutta method. We introduce for
\( t = nh \) (with \( n \) a positive integer) the Runge-Kutta map
\[
\mathcal{R}_h^t : C \times C \times C^m \times C^m \longrightarrow C \times C \times C^m \times C^m
\]
\[
(y^0, v^0, Y^0, V^0) \longrightarrow (y^t, v^t, Y^t, V^t)
\]
(with \( C^m = C \times \cdots \times C \) denoting the \( m \)-fold Cartesian product of \( C \)) as follows:
we set
\[
(y^0(0), v^0(0)) = (y_n, v_n),
\]
(4)
the numerical solution values at \( t = nh \). We define \((y^t(\theta), v^t(\theta)) (-h \leq \theta \leq 0)\) by
polynomial interpolation using as support values \((y_k, v_k)\) for \( k = n-p-1, \ldots, n-1 \)
(with the convention that, for negative \( k \), \((y_k, v_k) = (y^0(kh), v^0(kh))\)). Then we set recursively
\[
y^t(\theta) = y^{t-h}(\theta + h), \quad v^t(\theta) = v^{t-h}(\theta + h) \quad -\tau \leq \theta \leq -h.
\]
We construct the internal stages \( Y^t = \{Y_i^t\}_{i=1}^m \in C^m \) and \( V^t = \{V_i^t\}_{i=1}^m \in C^m \) by
\[
Y_i^t(\theta) = Y_i(t + \theta), \quad V_i^t(\theta) = V_i(t + \theta), \quad -\tau \leq \theta \leq 0,
\]
where \( Y_i(t) \) and \( V_i(t) \) are defined by
\[
\begin{aligned}
Y_i(t) &= y(t) + h \sum_{j=1}^{m} a_{ij} V_j(t) \\
0 &= V_i(t) - f (Y_i(t), Y_i(t - \tau)) - A V_i(t - \tau).
\end{aligned}
\]
(5)
For negative times \( t \in [-\tau, 0] \), the required values \((Y_i(t), V_i(t))\) are given by the
initial data \((Y_i^0, V_i^0)\). For initial data of the internal stages lying on the exact initial
The main result of this paper reads as follows.

Under the assumptions of Theorem 3.1, the Hausdorff distance of the numerical method.

3. Invariant closed curves of the numerical method.

3.1. Principal result. The main result of this paper reads as follows.

Main Theorem. Assume that (7) has a stable hyperbolic periodic orbit. Then, for sufficiently small stepsize $h$, there exist closed curves $A_h, B_h$ in $C \times C \times C^m \times C^m$ that are invariant under $R_h$ and $S_h$, respectively. $A_h$ attracts exponentially, uniformly in $h$, all functions in some neighborhood that have a fixed Lipschitz constant. Similarly, $B_h$ is locally exponentially attractive. The Hausdorff distance of the two invariant sets is bounded by

$$\text{dist}_H(A_h, B_h) = O(h^p),$$

where $p$ is the classical order of the Runge–Kutta method.

The projection of $B_h \subset C \times C \times C^m \times C^m$ onto the first two components is the periodic orbit of (7).

$$\Gamma = \{(x^t, 0^t) : t \in \mathbb{R}\} \subset C \times C.$$ We define its numerical analogue $\Gamma_h \subset C \times C$ as the projection of $A_h \subset C \times C \times C^m \times C^m$ onto the first two components. Theorem 3.1 yields the following corollary.

Corollary 1. Under the assumptions of Theorem 3.1 the Hausdorff distance between the numerical periodic orbit $\Gamma_h$ and the exact periodic orbit $\Gamma$ of the neutral delay differential equation (7) is bounded by

$$\text{dist}_H(\Gamma_h, \Gamma) = O(h^p).$$

$\Gamma_h$ is locally exponentially attractive in the sense that for any $L > 0$ there exist positive constants $h_0, \mu, M$ such that for any $h \leq h_0$ and for initial data $(x^0, u^0)$ that are Lipschitz continuous with constant $L$ and sufficiently close to the periodic solution, the interpolated Runge–Kutta solution $(y^t, v^t)$ satisfies

$$\text{dist}((y^t, v^t), \Gamma_h) \leq M e^{-\mu t} \quad \text{for all} \quad t = nh \geq 0.$$
3.2. Existence of the attractive invariant curves. The proof of existence of the invariant curves and of their $h$-uniform attractivity is not detailed here, since it follows similar lines to the corresponding result for the case of delay differential equations studied in [11]. It is based on an invariant manifold theorem applied to the maps $S_h^k$ and $R_h^k$ (for sufficiently large $t = nh$) written in normal coordinates, which are given as follows: Concerning the first component $x$, by [17] Theorem 5.7 A,B we obtain the existence of a decomposition of the space $C$,

$$C = E(\alpha) \oplus K(\alpha)$$

where $E(\alpha)$ is the eigenspace of $D_x S^\omega_x (\bar{x}^\alpha, \bar{u}^\alpha)$ related to the eigenvalue $\lambda = 1$, and $K(\alpha)$ is a closed linear subspace of $C$ invariant under $D_x S^\omega_x (\bar{x}^\alpha, \bar{u}^\alpha)$ such that the spectrum of the restriction is inside the unit disk:

$$\rho\left(D_x S^\omega_x (\bar{x}^\alpha, \bar{u}^\alpha) \big|_{K(\alpha)}\right) < 1.$$ 

Concerning the second component $u$, we keep the standard coordinates. In terms of these normal coordinates $(\alpha, \phi, u)$, it can be verified that $S_h^k$ satisfies the conditions of the invariant manifold theorem of [12]. As in [11], the construction of the invariant curve for $R_h^k$ uses $O(h)$ finite-time error bounds for initial data that have no smoothness other than Lipschitz continuity. With some technical complications, the proof from [11] carries over to the neutral situation considered here.

3.3. Distance of the invariant curves. A basic difference to the arguments used in the pure delay case arises in the proof of the distance estimate. In [11], that proof was based on the smoothing property of the delay differential equation which ensures that after an elapse of $k$ time delays, the solution is $k$ times continuously differentiable. Such a smoothing property is not available in the neutral case, and we need to develop a different argument to conclude to the full order of approximation $p$ in the distance estimate.

Let $(x^0, u^0, X_1^0, \ldots, X_m^0, U_1^0, \ldots, U_m^0)$ be on the invariant curve $B_h$ of $S_h$. Then $x^0$ is a segment of the $\omega$-periodic solution $x = \bar{x} : \mathbb{R} \to \mathbb{R}^d$, $u^0$ a segment of $u = \bar{u}$, and $X_i^0, U_i^0$ are segments of $\omega$-periodic functions $X_i, U_i : \mathbb{R} \to \mathbb{R}^d$ that satisfy (6). The key to proving the distance estimate is the following local error estimate.

**Lemma 1.** In the above situation of initial data on the invariant curve $B_h$ of $S_h$ we have, for all $t \in \mathbb{R}$,

$$x(t + h) - x(t) - h \sum_{i=1}^{m} b_i U_i(t) = O(h^{p+1}).$$

**Proof.** (a) Using Equation (1) recursively, we obtain the infinite-delay equation, cf. [15],

$$\dot{x}(t) = \sum_{k \geq 0} A^k f(x(t-k\tau), x(t-(k+1)\tau)). \quad (7)$$

Denoting the history of $x$ up to time $t$ by

$$x^{(t)} : (-\infty, 0] \to \mathbb{R}^d, \quad x^{(t)}(s) = x(t+s) \text{ for } s \leq 0,$$

we can rewrite (7) as

$$\dot{x}^{(t)} = F(x^{(t)})$$

with

$$F(x)(s) = \sum_{k \geq 0} A^k f(x(s-k\tau), x(s-(k+1)\tau))$$
for \( s \leq 0 \) and for any \( x \in BC \), where \( BC := BC((-\infty, 0], \mathbb{R}^d) \) is the Banach space of bounded continuous functions on the negative half-line, equipped with the maximum norm. The mapping

\[
F : BC \to BC
\]

is arbitrarily often differentiable (in the Fréchet sense) if \( f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \) is arbitrarily differentiable, as was assumed. The first derivative is given by

\[
[F'(x)u](s) = \sum_{k \geq 0} A^k \left( D_1 f(x(s-k\tau), x(s-(k+1)\tau)) u(s-k\tau) + D_2 f(x(s-k\tau), x(s-(k+1)\tau)) u(s-k\tau) \right)
\]

for \( s \leq 0 \). It is indeed straightforward to check that with this expression for \( F'(x)u \), we have in the maximum norm \( \|F(x+u) - F(x) - F'(x)u\| = O(\|u\|^2) \). Higher derivatives are obtained similarly.

(b) Using the last equation of (6) recursively, we obtain

\[
U_i^{(t)} = F(X_i^{(t)}),
\]

and \( X_i^{(t)} \) and \( U_j^{(t)} \) are related by the second equation of (6),

\[
X_i^{(t)} = x^{(t)} + h \sum_{j=1}^m a_{ij} U_j^{(t)}, \quad i = 1, \ldots, m.
\]

We now compare the Taylor expansions, in powers of \( h \), of \( x^{(t+h)} \) and of

\[
x_h^{(t)} := x^{(t)} + h \sum_{i=1}^m b_i U_i^{(t)}.
\]

The basic observation is now that these expansions are formally the same as for ordinary differential equations \( y' = f(y) \), where they are represented by trees and elementary differentials; see, e.g., [1] or [8] Chap. III. We have

\[
\frac{d^2 x^{(t)}}{dt^2} = F'(x^{(t)}) \frac{dx^{(t)}}{dt} = F'(x^{(t)}) F(x^{(t)}).
\]

Similarly we have the higher time derivatives

\[
\frac{d^3 x^{(t)}}{dt^3} = F'(x^{(t)}) F'(x^{(t)}) F(x^{(t)}) + F''(x^{(t)}) \{ F'(x^{(t)}), F'(x^{(t)}) \},
\]

\[
\frac{d^4 x^{(t)}}{dt^4} = F'' F' F + F'' F'' (F, F) + 3 F'' F' F + F'' (F, F, F), \quad \text{etc.}
\]

In the same way, the derivatives of \( x_h^{(t)} \) with respect to \( h \) at \( h = 0 \) are linear combinations of the elementary differentials \( F, F' F, F' F' F, F''(F, F), \ldots \) evaluated at \( x^{(t)} \), with the same coefficients that appear in the Taylor expansion of the Runge-Kutta method applied to ordinary differential equations. Therefore, all terms up to and including \( O(h^p) \) in the Taylor expansions of \( x^{(t+h)} \) and \( x_h^{(t)} \) coincide. It follows that

\[
\|x_h^{(t)} - x^{(t+h)}\| = O(h^{p+1})
\]

where the norm is the maximum norm on \((-\infty, 0] \). Evaluated at \( s = 0 \), this gives the stated result. \( \square \)
Lemma 1 yields that for initial data $z^0 = (x^0, u^0, X^0_1, \ldots, X^0_m, T^0_1, \ldots, T^0_m) \in B_h$ we have the local error bound

$$R_h(z^0) - S_h(z^0) = O(h^{p+1}).$$

To obtain error bounds on finite time intervals, we need the stability of the Runge-Kutta map $R_h$: there is a constant $L$ such that for $z^0, w^0 \in C \times C \times C^m \times C^m$,

$$\|R_h(z^0) - R_h(w^0)\| \leq e^{hL} \|z^0 - w^0\|.$$  

Taken together, the above two properties yield the following.

**Lemma 2.** For $z^0 = (x^0, u^0, X^0_1, \ldots, X^0_m, U^0_1, \ldots, U^0_m) \in B_h$ and $t = nh \leq \text{Const.},$ we have

$$R^t_h(z^0) - S^t_h(z^0) = O(h^p).$$

However, such an error bound does not hold in any open neighborhood of $B_h$. Since $R^t_h$ and $S^t_h$ satisfy, when rewritten in normal coordinates and for sufficiently large $t$, the conditions of the attractive invariant manifold theorem of Kirchgraber, Lasagni, Nipp and Stoffer [12] (see also [8, Chap. XII]), the $O(h^p)$ distance estimate between the invariant manifolds of $R^t_h$ and $S^t_h$ now follows from a corollary to the invariant manifold theorem as given in [11] (or again in [8, Chap. XII]) which requires closeness of the maps only on the invariant manifold of one of the maps, here on the invariant curve $B_h$ of $S^t_h$.

4. **Numerical experiments.**

**The methods.** We consider the following explicit Runge–Kutta methods:
- Method 1: the Heun method of order 3 [9, p. 135],

**Numerical verification.** We are interested in the trajectories in the $(x, u)$-plane (where $u = x'$). We aim to give experimental confirmation that the numerical solution of (I) has an invariant curve $\Gamma_h$ approximating the periodic orbit $\Gamma$ of (I) (when it exists) and is such that

$$\text{dist}_{\mathcal{H}}(\Gamma_h, \Gamma) = O(h^p).$$

**Problem 1.** We consider the following problem (linear in the neutral term) arising from a model of lossless transmission lines (see e.g. [14, 18]),

$$\begin{cases}
  x'(t) &= \rho x'(t - \tau) - ax(t) - \rho bx(t - \tau) + g(x(t) - \rho x(t - \tau)) & t \geq 0 \\
  x^0(t) &= 1 + t & t \leq 0
\end{cases} \quad (8)$$

with $g(z) = z^3$.

For a suitable choice of the parameters, determining hyperbolic periodic orbits of the equation, we make use of [18] Theorem 3.6. In particular, according to this result, we consider the following choice:

$$a = 1 \quad b = 4 \quad \rho = \frac{9}{10} \quad \tau = \frac{1}{2}.$$  

which corresponds to a unique stable periodic orbit.

The following table illustrates the computational results obtained by applying both methods considered. For large stepsizes $h = \tau/\nu$ (say $\nu \leq 10$) the numerical solution diverges for both the considered explicit methods, due to stability restrictions.
The periodic orbit is actually computed by means of the code RADAR5 (7) to a very high accuracy \(10^{-10}\). Table 1 shows the (computed) Hausdorff distance between the periodic orbit \(\Gamma\) and the orbits \(\Gamma_h\) computed through the considered methods with step size \(h = 1/\nu\).

<table>
<thead>
<tr>
<th>(\nu)</th>
<th>Method 1</th>
<th>(\nu)</th>
<th>Method 2</th>
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<td>(h) (\text{dist}_H)</td>
<td></td>
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<td>12</td>
<td>(\frac{1}{24}) (9.24936e-4)</td>
<td>16</td>
<td>(\frac{1}{32}) (3.84495e-5)</td>
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<tr>
<td>16</td>
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<td>18</td>
<td>(\frac{1}{36}) (1.20154e-6)</td>
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<td>32</td>
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<td>(\frac{1}{40}) (5.77969e-7)</td>
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<td>64</td>
<td>(\frac{1}{128}) (5.30726e-6)</td>
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<td>(\frac{1}{44}) (3.32298e-7)</td>
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<tr>
<td>128</td>
<td>(\frac{1}{256}) (6.89843e-7)</td>
<td>24</td>
<td>(\frac{1}{48}) (2.19992e-7)</td>
</tr>
<tr>
<td>256</td>
<td>(\frac{1}{512}) (1.56358e-7)</td>
<td>26</td>
<td>(\frac{1}{52}) (1.52682e-7)</td>
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</table>

**Table 1.** Statistics for the problem (8) for the two considered methods.

The work-precision diagram related to the two methods in Table 1 is the following.

![Diagram](image)

**Figure 1:** Diagram (in logarithmic scale) illustrating the Hausdorff distance between the true and the numerical periodic orbits computed by both methods \((M_1, M_2)\), for different choices of the stepsize.

Figure 2 shows both the periodic orbit \(\Gamma\) and a subset of consecutive approximating values \((y_n, v_n)\) computed by the numerical method of order \(p = 5\) (with stepsize \(h = 1/32\)).

The experimentally estimated orders of the two methods are \(p = 2.81670\ldots\) and \(p = 4.84710\ldots\) respectively (see Figure 1).
Problem 2. We consider the logistic-like model problem (nonlinear in the neutral term)
\[
\begin{cases}
    x'(t) &= x(t) \left( \alpha - x(t-1) - \rho x'(t-1) \right) & t \geq 0 \\
    x^0(t) &= 0.33 - \frac{t}{10} & t \leq 0
\end{cases}
\]
with \(\alpha, \rho\) suitable constants determining — as far as the numerical evidence suggests — a periodic solution of the equation. We applied the two explicit Runge–Kutta methods to the problem (9) for several choices of the parameters \((\alpha, \rho)\). In all cases we were able to verify Corollary 1. We illustrate the numerical results obtained in two of such cases.

Case 1: \((\alpha, \rho) = (2.0, 0.2)\). The periodic orbit has an oval shape. It is computed by means of the code RADAR5 to a very high accuracy (see the left picture in Figure 4).

Case 2: \((\alpha, \rho) = (2.0, 0.7)\). The periodic orbit in this case has a more articulated shape, with a cross point (see the right picture in Figure 4).

Table 2 shows the computed Hausdorff distance.

For case 1 the experimentally estimated orders for the two methods are \(p = 3.0135\ldots\) and \(p = 5.0989\ldots\) respectively (see Figure 3). For case 2 the estimated orders, \(p = 3.0143\ldots\) and \(p = 5.2622\ldots\) respectively, are very close to the previous ones.

We observe again that for large stepsizes the numerical solution diverges, due to stability restrictions for the considered explicit methods.
<table>
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<th>$\text{dist}_H$</th>
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<td>1.883e - 2</td>
<td>1.741e - 4</td>
</tr>
<tr>
<td>8</td>
<td>2.461e - 3</td>
<td>3.135e - 6</td>
</tr>
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<td>2.982e - 4</td>
<td>5.707e - 8</td>
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<td>0</td>
</tr>
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</table>

Table 2. Statistics for the problem (9). Case 1 (left side); case 2 (right side)

The diagram related to Table 2 follows.

Figure 3: Diagram illustrating the Hausdorff distance between the periodic orbit and the numerical invariant curve for different choices of the stepsize $h = 1/\nu$.

The following picture shows the periodic orbits.

Figure 4: Periodic orbits of (9) represented in the $(x, u)$-plane for the parameter pairs $(\alpha, \rho) = (2.0, 0.2)$ (left) and $(2.0, 0.7)$ (right).
In conclusion, the experiments indicate that for both kinds of model problems, that is in the case of a linear dependence on the neutral term (Problem 1) and in the case of a nonlinear dependence (Problem 2), there is a numerical periodic orbit with \( \text{dist}_N(\Gamma_h, \Gamma) = O(h^p) \).

**Numerical routines.** Both the code RADAR5 and the routines used in this work are available from the address http://univaq.it/~guglielm/.

Functions are written in the Matlab 6.1 environment. The available functions refer to Problem 2.

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**REFERENCES**


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